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WENTWORTH'S SERIES OF MATHEMATICS.

A COLLEGE ALGEBRA.

BY

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PREFACE.

THIS work, as the name implies, is intended for Colleges and Scientific Schools. The first part is simply a review of the principles of Algebra preceding Quadratic Equations, with just enough examples to illustrate and enforce these principles. By this brief treatment of the first chapters, sufficient space is allowed, without making the book cumbersome, for a full discussion of Quadratic Equations, The Binomial Theorem, Choice, Chance, Series, Determinants, and The General Properties of Equations. Every effort has been made to present in the clearest light each subject discussed, and to give in matter and methods the best training in algebraic analysis at present attainable. The work is designed for a full-year course. Sections and problems marked with a star can be omitted, if necessary; and for a half-year course many chapters must be omitted.

The author gratefully acknowledges his obligation to Mr. G. W. Sawin of Harvard College, who has contributed the excellent chapter on Determinants, and been of invaluable assistance in revising every chapter of the book.

Answers to the problems are bound separately in paper covers, and will be furnished free to pupils when *teachers* apply to the publishers for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
September, 1888.

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COLLEGE ALGEBRA.

CHAPTER I.

FUNDAMENTAL IDEAS.

1. Quantity and Number. Whatever may be regarded as being made up of parts like the whole is called a **quantity.**

In other words, whatever admits of division into parts all *the same in kind* as the whole is a *quantity.*

To **measure** a quantity of any kind is to find how many times it contains another *known quantity of the same kind.*

A *known quantity* which is adopted as a standard for measuring quantities of the same kind is called a **unit.**

Thus, the foot, the pound, the dollar, the day, are units for measuring distance, weight, money, time.

A **number** arises from the repetitions of the unit of measure, and shows *how many times* the unit is contained in the quantity measured.

2. When a quantity is measured, the result obtained is expressed by prefixing to the *name* of the unit the *number* which shows how many times the unit is contained in the quantity measured.

This result is called the **measure** of the quantity. The number which shows how many times the unit is taken is called the **numerical measure** of the quantity.

Thus, 7 feet, 8 pounds, 9 dollars, 14 days, are respectively measures of a distance, a weight, an amount of money, and an interval of time; the numerical measures being respectively the numbers 7, 8, 9, and 14.

3. For convenience, numbers are represented by symbols. In Arithmetic the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and combinations of these symbols, are employed to represent numbers. The series 0, 1, 2, 3, ..., obtained by counting, is called the *natural series* of numbers.

Any figure or combination of figures represents one, and but one, particular number.

4. Numbers in General. Numbers possess many *general properties*, which are true, not only of a particular number, but of all numbers.

For example, the sum of 12 and 8 is 20, and the difference between 12 and 8 is 4. Their sum added to their difference is 24, which is twice the greater number. Their difference taken from their sum is 16, which is twice the smaller number.

We shall see later on that these are general properties of numbers, namely:

The sum of two numbers added to their difference is twice the greater number; the difference of two numbers taken from their sum is twice the smaller number. Or,

(1) (greater number + smaller number) + (greater number − smaller number) = twice greater number.

(2) (greater number + smaller number) − (greater number − smaller number) = twice smaller number.

But these statements may be very much shortened; for, as greater number and smaller number may mean any two numbers, two letters, as a and b, may be used to represent

them; then $2a$ will represent twice the greater number, and $2b$ twice the smaller. Then these statements become:

$$(1)\ (a+b)+(a-b)=2a.$$
$$(2)\ (a+b)-(a-b)=2b.$$

In studying the general properties of numbers, letters used to represent numbers may represent any numerical values consistent with the conditions of the problem.

5. Algebra like Arithmetic is a science which treats of *numbers*. In any problem in which we are concerned with *quantities*, we use not the quantities themselves, but their *numerical measures.*

In Algebra as in Arithmetic we use the Arabic numerals to represent particular numbers. But in Algebra we also use other symbols, generally the letters of the alphabet, to represent numbers.

Algebra is, then, a species of generalized Arithmetic, and includes the ordinary Arithmetic.

6. Operations to be performed upon numbers are indicated in Algebra, as in Arithmetic, by **signs.**

The chief signs of operation used in Arithmetic are the following:

$+$ (read, *plus*), the sign of addition.
$-$ (read, *minus*), the sign of subtraction.
$\times$ (read, *multiplied by*), the sign of multiplication.
$\div$ (read, *divided by*), the sign of division.

7. Positive and Negative Numbers. There are quantities which stand to each other in such opposite relations that, when we combine them, they cancel each other entirely or in part. Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and leave four dollars gain.

An opposition of this kind exists in *assets* and *debts*, in motion *forwards* and motion *backwards*, in motion *to the right* and motion *to the left*, in the degrees *above* and the degrees *below* zero on a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards*, along the natural series of numbers. If we wish to substract four from six, we start at six in the natural series, count four units backwards, and arrive at two, the difference sought. If we subtract six from six, we start at six in the natural series, count six units backwards, and arrive at zero. If we try to subtract nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers comes to an end.*

8. In order to subtract a greater number from a smaller, it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must ascend from zero by the repetitions of the unit, precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign $+$; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign $-$. The two series of numbers will be written thus:

$$\cdots\cdots -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ \cdots\cdots$$

If, now, we wish to subtract 9 from 6, we begin at 6 in the positive series, count nine units in the *negative direction*

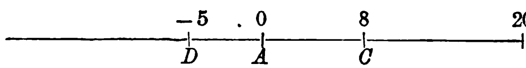
(to the left), and arrive at -3 in the negative series. That is, $6-9=-3$.

The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number*.

If a and b represent any two numbers of the positive series, the expression $a-b$ will denote a positive number when a is greater than b; will be equal to zero when a is equal to b; will denote a negative number when a is less than b.

If we wish to add 9 to -6, we begin at -6, in the negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$, in the positive series.

We may illustrate the use of positive and negative numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? Answer: at C, a point 8 feet to the right of A. That is, 20 feet − 12 feet = 8 feet; or, $20-12=8$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D, a point 5 feet to the left of A. That is, if we consider distance measured in feet to the left of A as forming a negative series of numbers, beginning at A, $20-25=-5$. Hence, the phrase, 5 feet to the left of A, is now expressed by the negative number -5.

9. Numbers with the sign + or − are called **algebraic numbers.** They are unknown in Arithmetic, but play a very important part in Algebra. Numbers not affected by the signs + or − are called **absolute numbers.**

Every algebraic number, as $+4$ or -4, consists of a sign + or − and the absolute value of the number; in this case 4. The sign shows whether the number belongs

to the positive or negative series of numbers; the absolute value shows what place the number has in the positive or negative series.

When no sign stands before a number, the sign $+$ is always understood; thus, 4 means the same as $+4$, a means the same as $+a$. But *the sign $-$ is never omitted.*

Two numbers which have, one the sign $+$ and the other the sign $-$, are said to have **unlike signs.**

Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined; thus, $+4-4=0$, $+a-a=0$.

10. Meaning of the Signs. The use of the signs $+$ and $-$, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense, they are signs of *operations*, and are common to both Arithmetic and Algebra. In the second sense, they are signs of *opposition*, and are employed in Algebra alone.

11. Factors. When a number consists of the product of two or more numbers, each of these numbers is called a **factor** of the product.

When these numbers are denoted by letters, the sign $\times$ is often omitted; thus, instead of $a \times b$, we write ab; instead of $a \times b \times c$, we write abc.

Factors expressed by letters are called **literal** factors; factors expressed by figures are called **numerical** factors.

12. A known factor of a product which is prefixed to another factor to show how many times that factor is taken is called a **coefficient.**

13. Powers. A product consisting of two or more **equal** factors is called a **power** of that factor.

The **index** or **exponent of a power** is a small figure or letter placed at the right of a number, to show how many times the number is taken as a factor.

Thus, a^3 is written instead of *aaa.*

a^n is written instead of *aaa* ..... to n factors.

The second power of a number is generally called the *square* of that number; the third power of a number, the *cube* of that number.

14. Signs. The principal signs used in Algebra in addition to those of § 6 are the following:

The signs of **relation**: $=$, $>$, $<$, which stand for *is equal to, is greater than,* and *is less than,* respectively.

The signs of **aggregation**: the bar, |; the vinculum, ——; the parenthesis, ( ); the bracket, [ ]; and the brace, { }.

Thus, each of the expressions, $\left.\begin{matrix}x\\+y\end{matrix}\right|$, $\overline{x+y}$, $(x+y)$, $[x+y]$, $\{x+y\}$, signifies that $x+y$ is to be treated as a single number.

The signs of **continuation**: dots, ..., or dashes, – – – – –, read, *and so on.*

The sign of **deduction**: $\therefore$, read, *hence,* or *therefore.*

Remark. When a sign of operation is omitted between numerals it is the sign of *addition;* when between letters, or a numeral and a letter, it is the sign of *multiplication.* Thus, 423 means $400+20+3$, but $2abc$ means $2\times a\times b\times c$.

15. An **algebraic expression** is a number written with algebraic symbols; an algebraic expression consists of one symbol, or of several symbols connected by signs of operation.

A **term** is an algebraic expression the parts of which are not separated by the sign of addition or subtraction. Thus, $3ab$, $5xy$, $3ab\div 4xy$ are terms.

A **monomial** or **simple expression** is an expression with but one term.

A **polynomial** or **compound expression** is an expression of

two or more terms. A **binomial** is a polynomial of two terms; a **trinomial**, a polynomial of three terms.

Like terms or **similar terms** are terms which have the same letters, and the corresponding letters affected by the same exponents. Thus, $7a^2cx^3$ and $-5a^2cx^3$ are like terms.

16. The **degree** of a term is the sum of the exponents of its literal factors. Thus, $3xy$ is of the *second* degree, and $5x^2yz^3$ of the *sixth* degree.

A polynomial is said to be **homogeneous** when all its terms are of the same degree. Thus, $7x^3 - 5x^2y + xyz$ is homogeneous of the third degree.

A polynomial is said to be **arranged** according to the powers of some letter when the exponents of that letter either descend or ascend in order of magnitude.

17. The **value** of an algebraic expression is the number which the expression represents.

If the number represented by each symbol involved in an expression is known, the value of the expression can be found by putting for each symbol the number it represents and performing the indicated operations.

The value of an expression evidently depends upon the values given to the several symbols involved.

18. Axioms. 1. Things which are equal to the same thing are equal to each other.

2. If equal numbers be added to equal numbers, the sums will be equal numbers.

3. If equal numbers be subtracted from equal numbers, the remainders will be equal numbers.

4. If equal numbers be multiplied into equal numbers, the products will be equal numbers.

5. If equal numbers be divided by equal numbers, the quotients will be equal numbers.

CHAPTER II.

FUNDAMENTAL OPERATIONS. — ADDITION.

19. An algebraic number which is to be added or subtracted is often inclosed in a parenthesis, in order that the signs $+$ and $-$ which are used to distinguish positive and negative numbers may not be confounded with the $+$ and $-$ signs that denote the operations of addition and subtraction. Thus, $+4+(-3)$ expresses the sum, and $+4-(-3)$ expresses the difference, of the numbers $+4$ and -3.

20. Monomials. In order to add two algebraic numbers, we begin at the place in the series which the first number occupies, and count, *in the direction indicated by the sign of the second number*, as many units as there are units in the absolute value of the second number. Thus, the sum of $+4+(+3)$ is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$; the sum of $+4+(-3)$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$.

In like manner, the sum of $-4+(+3)$ is -1, and the sum of $-4+(-3)$ is -7.

I. Therefore, to add two numbers with *like* signs, find the *sum* of their absolute values, and prefix the common sign to the sum.

II. To add two numbers with *unlike* signs, find the *difference* of their absolute values, and prefix the sign of the number absolutely greater to the difference. Thus,

(1) $+a+(+b)=a+b$; (3) $-a+(+b)=-a+b$;

(2) $+a+(-b)=a-b$; (4) $-a+(-b)=-a-b$.

21. It should be noticed that the *order* of the terms is immaterial. Thus, $+a-b=-b+a$. This law is called the **commutative law** for addition.

22. By successive application of the above rules we readily obtain rules for adding any number of terms.

Thus,
$$4a+5a+3a+2a=14a;$$
$$-3a-15a-7a+14a-2a=14a-27a=-13a;$$
$$4a-3b-9a+7b=-5a+4b.$$

23. Polynomials. Two or more polynomials are added by adding their separate terms.

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column. Thus,

$$\begin{array}{rrrr}
2a^3 & -3a^2b & +4ab^2 & +\ b^3 \\
a^3 & +4a^2b & -7ab^2 & -2b^3 \\
-3a^3 & +\ a^2b & -3ab^2 & -4b^3 \\
2a^3 & +2a^2b & +6ab^2 & -3b^3 \\
\hline
2a^3 & +4a^2b & & -8b^3
\end{array}$$

Exercise 1.

Add:

1. $9a^2+3a+4b,\ 2a^2-4a+5b,\ 5a-2b-6a^2.$
2. $7x^2-2xy+y^2,\ 4xy-2y^2,\ 8x^2-9xy+12y^2.$
3. $7a^2b+9ab^2-13b^3,\ 3a^3+2ab^2-7b^3,$
   $ab^2-a^2b-6a^3,\ 5b^3-7a^3-ab^2,\ 4b^3-2a^3+a^2b.$
4. $5x^4+2x^2-7,\ 4x^3+x-9,\ 1+x-x^2,$
   $x^5+x^4-x^3-x^2-7,\ 9x^2+9x^3-12x-4x^4+10.$
5. $3m^4+2m^3n+5m^2n^2-9n^4,\ 7n^4-3mn^3-8m^2n^2,$
   $11mn^3-4m^2n^2+6m^3n,\ 5m^4+2m^3n-15mn^3-7n^4.$
6. $2x^6+3x^5y-4x^4y^2,\ 2y^6-3xy^5+4x^2y^4-10x^3y^3,$
   $5x^3y^3+4x^2y^4-9y^6,\ 8x^5y-7x^4y^2+6x^3y^3-8x^2y^4.$

SUBTRACTION.

24. Monomials. In order to find the difference between two algebraic numbers, we begin *at the place in the series which the minuend occupies*, and *count in the direction opposite to that indicated by the sign of the subtrahend* as many units as there are units in the absolute value of the subtrahend.

Thus, the difference between $+4$ and $+3$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$; the difference between $+4$ and -3 is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$.

In like manner, the difference between -4 and $+3$ is -7; the difference between -4 and -3 is -1.

Compare these results with results obtained in addition; it is evident that:

Subtracting a *positive* number is equivalent to adding an equal *negative* number.

Subtracting a *negative* number is equivalent to adding an equal *positive* number.

To subtract, therefore, one algebraic number from another, *change the sign of the subtrahend, and then add it to the minuend.*

Thus,

$$+a-(+b)=a-b; \qquad -a-(+b)=-a-b;$$
$$+a-(-b)=a+b; \qquad -a-(-b)=-a+b.$$

25. Polynomials. When one polynomial is to be subtracted from another, place its terms under the like terms of the other, change the signs of the subtrahend, and add.

$$\begin{array}{ll} \text{From} & 4x^3-3x^2y-\ \ xy^2+2y^3 \\ \text{take} & \underline{2x^3-\ \ x^2y+5xy^2-3y^3} \end{array}$$

Change the signs of the subtrahend and add:

$$\begin{array}{r} 4x^3 - 3x^2y - xy^2 + 2y^3 \\ -2x^3 + x^2y - 5xy^2 + 3y^3 \\ \hline 2x^3 - 2x^2y - 6xy^2 + 5y^3 \end{array}$$

Instead of actually changing the signs of the subtrahend we need only *conceive* them to be changed.

26. Parentheses. From (§ 24), it appears that

(1) $a + (+b) = a + b$. (3) $a - (+b) = a - b$.

(2) $a + (-b) = a - b$. (4) $a - (-b) = a + b$.

The same laws respecting the removal of parenthesis hold true whether one or more terms are inclosed. Hence, when an expression within a parenthesis is preceded by a *plus sign*, the parenthesis may be removed.

When an expression within a parenthesis is preceded by a *minus sign*, the parenthesis may be removed *if the sign of every term within the parenthesis is changed.*

Thus, (1) $a + (b - c) = a + b - c$.

(2) $a - (b - c) = a - b + c$.

27. Observe that the terms may be combined in any manner.

Thus,
$$\begin{aligned} a + b - c - d &= (a + b) - (c + d) \\ &= (a + b - c) - d \\ &= a + (b - c - d). \end{aligned}$$

This is called the **associative law** for addition and subtraction.

28. Expressions often occur with more than one parenthesis. These parentheses may be removed in succession, by removing *first, the innermost parenthesis;* next, the innermost of all that remain, and so on.

Thus,
$$
\begin{aligned}
&a-[b-\{c+(d-\overline{e-f})\}]\\
&=a-[b-\{c+(d-e+f)\}]\\
&=a-[b-\{c+d-e+f\}]\\
&=a-[b-c-d+e-f]\\
&=a-b+c+d-e+f.
\end{aligned}
$$

29. The rules for introducing parentheses follow directly from the rules for removing them:

1. Any number of terms of an expression may be put within a parenthesis, and the sign $+$ placed before the whole.

2. Any number of terms of an expression may be put within a parenthesis, and the sign $-$ placed before the whole; *provided the sign of every term within the parenthesis be changed.*

Exercise 2.

1. From $4a+5b-3c$ take $2a+9b-8c$.

2. From $7x^3-x^2+4x-2$ take $2x^3+8x^2-9x+8$.

3. From $3a^3+3a^2b-9ab^2+3b^3$
take $2a^3-5a^2b+7ab^2-9b^3$.

4. From $\frac{1}{3}ab+4a^2-\frac{2}{5}b^2+\frac{1}{4}a$ take $a^2-\frac{1}{10}b^2+\frac{1}{8}a$.

5. From $4x^3-6x^2+8x-7$ take the sum of
$8x^3+7-8x^2+7x$ and $-9x^3-8x^2+4x+4$.

Simplify:

6. $2-3x-(4-6x)-\{7-(9-2x)\}$.

7. $3a-(a-b-c)-2\{a+c-2(b-c)\}$.

8. $4a-[3a-\{2a-(a-b)\}+5b]$.

9. $[8a-3\{a-(b-a)\}]-4[a-2\{a-2(a-b)\}+b]$.

10. $x(y+z)+y[x-(y+z)]-z[y-x(z-x)]$.

11. $2x^3(x-3a)-2[2x^4-a^2(x^2-a^2)]$
$-3a[x^3-2x\{a^2+x(a-x)\}+a^3]$.

MULTIPLICATION.

30. Let a and b be any two members. To obtain the product of a by b we do to a what we do to unity to obtain b.

Thus, to obtain 5 we take 1 five times, and to obtain the product 5×3 we take 3 five times.

Similarly, to obtain -5 we take 1 five times, and then change the sign of the product; hence, to obtain the product $(-5) \times (-3)$ we take -3 five times, giving -15, and then change the sign, giving $+15$.

In general,

$$a \times b = +ab; \qquad (-a) \times b = -ab;$$
$$a \times (-b) = -ab; \qquad (-a) \times (-b) = +ab.$$

From the preceding we obtain the rule: *like* signs give *plus; unlike* signs give *minus.*

The product of more than two factors, each preceded by the sign $-$, will be *positive* or *negative*, according as the number of such factors is *even* or *odd.*

31. Monomials. The product of numerical factors is a new number in which no trace of the original factors is found. Thus, $4 \times 9 = 36$. But the product of literal factors is expressed by writing them one after the other. Thus, the product of a and b is expressed by ab; the product of ab and cd is expressed by $abcd$.

The product is evidently the same in whatever order the factors be written. This is the **commutative law** for multiplication.

32. Index Law. The product of two or more *powers* of any number is that number with an exponent equal to the *sum* of the exponents of the several factors.

For,

$$a^m \times a^n = (aaa \ldots\ldots \text{ to } m \text{ factors})(aaa \ldots\ldots \text{ to } n \text{ factors})$$
$$= aaaaaa \ldots\ldots \text{ to } (m+n) \text{ factors}$$
$$= a^{m+n}.$$

Similarly for more than two factors.

This law is called the **index law**.

33. The product of three or more factors is evidently the same in whatever way the factors be combined. Thus, $abcde = (abc) \times (de) = (ab) \times (cde)$, etc. This is the **associative law** for multiplication.

34. **Polynomials by Monomials.** If we have to multiply $a + b$ by n, that is, to take $(a + b)$ n times, we have,

$$(a+b) \times n = (a+b) + (a+b) + (a+b) \ldots\ldots n \text{ times},$$
$$= a + a + a \ldots\ldots n \text{ times} + b + b + b \ldots\ldots n \text{ times},$$
$$= a \times n + b \times n,$$
$$= an + bn.$$

As it is immaterial in what order the factors are taken,

$$n \times (a+b) = an + bn.$$

In like manner,

$$(a+b+c) \times n = an + bn + cn,$$

or,

$$n(a+b+c) = an + bn + cn.$$

The above is called the **distributive law** for multiplication.

35. **Polynomials by Polynomials.** If we have $a + b + c$ to be multiplied by $m + n + p$, we find,

$$(a+b+c)(m+n+p)$$
$$= (a+b+c)m + (a+b+c)n + (a+b+c)p$$
$$= am + bm + cm + an + bn + cn + ap + bp + cp.$$

In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

(1) Multiply $5a - 6b$ by $3a - 4b$.

$$\begin{array}{rrr}
5a & -6b & \\
3a & -4b & \\
\hline
15a^2 & -18ab & \\
 & -20ab & +24b^2 \\
\hline
15a^2 & -38ab & +24b^2
\end{array}$$

(2) Multiply $a^2 + b^2 + c^2 - ab - bc - ac$ by $a + b + c$.

Arrange according to descending powers of a.

$$\begin{array}{lllllllll}
a^2 & -ab & -ac & +b^2 & -bc & +c^2 & & & \\
a & +b & +c & & & & & & \\
\hline
a^3 & -a^2b & -a^2c & +ab^2 & -abc & +ac^2 & & & \\
 & +a^2b & & -ab^2 & -abc & & +b^3 & -b^2c & +bc^2 \\
 & & +a^2c & & -abc & -ac^2 & & +b^2c & -bc^2 & +c^3 \\
\hline
a^3 & & & & -3abc & & +b^3 & & & +c^3
\end{array}$$

Observe that, with a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier are arranged in the *same order*.

36. Detached Coefficients. In multiplying two polynomials which involve but one letter, or are homogeneous (§ 16) and involve but two letters, we shall save much labor if we write only the coefficients. Thus,

(1) Multiply $2x^3 + 4x + 7$ by $x^2 - 3x + 4$.

Since the x^2 term in the first expression is missing, we supply a zero coefficient. The work is as follows:

$$\begin{array}{rrrrrr}
2 & +0 & +4 & +7 & & \\
1 & -3 & +4 & & & \\
\hline
2 & +0 & +4 & +7 & & \\
 & -6 & -0 & -12 & -21 & \\
 & & +8 & +0 & +16 & +28 \\
\hline
2 & -6 & +12 & -5 & -5 & +28
\end{array}$$

Writing in the powers of x, the product is

$$2x^5 - 6x^4 + 12x^3 - 5x^2 - 5x + 28.$$

(2) Multiply $a^3 + 2ax^2 - 9x^3 + 4a^2x$ by $x^2 - 2ax - a^2$.

Arranging by powers of x we have

$$-9x^3 + 2ax^2 + 4a^2x + a^3 \text{ and } x^2 - 2ax - a^2.$$

The work is as follows:

$$\begin{array}{rrrrrr}
-9 & +\ 2 & +4 & +1 & & \\
1 & -\ 2 & -1 & & & \\
\hline
-9 & +\ 2 & +4 & +1 & & \\
 & +18 & -4 & -8 & -2 & \\
 & & +9 & -2 & -4 & -1 \\
\hline
-9 & +20 & +9 & -9 & -6 & -1
\end{array}$$

Hence, the product is

$$9x^5 + 20ax^4 + 9a^2x^3 - 9a^3x^2 - 6a^4x - a^5.$$

37. Special Cases. The following products are of great importance, and should be carefully remembered.

$$(a + b)^2 = a^2 + 2ab + b^2;$$
$$(a - b)^2 = a^2 - 2ab + b^2;$$
$$(a + b)(a - b) = a^2 - b^2;$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

The *square* of any polynomial may be immediately written down by the following rule: *Add together the squares of the several terms and twice the product of each term into each of the terms that follow it.*

Also:

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3;$$
$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4;$$

and so on.

38. Again consider the product

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

The coefficient of x is the *algebraic sum* of a and b; the third term is the *product* of a and b.

Thus,
$$(x+3)(x+7)=x^2+10x+21;$$
$$(x-3)(x+7)=x^2+\ 4x-21;$$
$$(x+3)(x-7)=x^2-\ 4x-21;$$
$$(x-3)(x-7)=x^2-10x+21.$$

Exercise 3.

Find the product of:

1. $3x+2y$ and $4x-5y$.
2. $2x^2-5$ and $4x+3$.
3. $2x^2+4x-3$ and $2x^2+3x-4$.
4. x^4+2x^2+4 and x^4-2x^2+4.
5. $x^2+2xy-3y^2$ and $x^2-5xy+4y^2$.
6. $9x^2+3xy+y^2-6x+2y+4$ and $3x-y+2$.
7. $11a^3+4b^3-4ab(a-4b)$ and $a^2(b+3a)-4b^2(a+b)$.
8. $(a+b)^2+(a-b)^2$ and $(a+b)^2-(a-b)^2$.
9. $x-2y+3z$ and $x-2y+3z$.
10. x^3+2x^2-4x-1 and x^3+2x^2-4x-1.
11. $39d^{x+y-1}-54d^{x-2y+1}+60d^{x+3y}$ and $30d^{2-x+2y}$.
12. $24x^{m+2n-1}-42x^{2m-3n+2}+25x^{2n+3m-2}$ and $25x^{2-m-2n}$.
13. $a^p-3a^{p-1}+4a^{p-2}-6a^{p-3}+5a^{p-4}$ and $2a^3-a^2+a$.
14. $a^{2n+1}-a^{n+1}-a^n+a^{n-1}$ and $a^{n+2}-a^2-a+1$.
15. $a^p+3a^{p-2}-2a^{p-1}$ and $2a^{p+1}+a^{p+2}-3a^p$.

DIVISION.

39. Division is the operation by which, when a product and one of its factors are given, the other factor is determined.

With reference to this operation the product is called the **dividend**; the given factor the **divisor**; and the required factor the **quotient**.

The operation of division is indicated by the sign $\div$; by the colon :, or by writing the dividend over the divisor with a line drawn between them. Thus, $12 \div 4$, $12 : 4$, $\frac{12}{4}$, each means that 12 is to be divided by 4.

40. Since $a \times b = +ab$; $(-a) \times b = -ab$;

$a \times (-b) = -ab$; $(-a) \times (-b) = +ab$;

therefore $\frac{ab}{b} = a$; $\frac{-ab}{+b} = -a$;

$\frac{-ab}{-b} = +a$; $\frac{+ab}{-b} = -a$.

Consequently, the quotient is *positive* when the dividend and divisor have *like* signs.

The quotient is *negative* when the dividend and divisor have *unlike* signs.

41. Monomials. To divide one monomial by another.

Write the dividend over the divisor with a line between them; if the expressions have common factors, remove the common factors.

Thus, $\frac{25\,abx}{10\,bcx} = \frac{5\,a}{2\,c}$; $\frac{36\,bcx}{30\,abc} = \frac{6\,x}{5\,a}$.

Again, $\dfrac{a^5}{a^2} = \dfrac{aaaaa}{aa} = aaa = a^3.$

$$\frac{a^2}{a^5} = \frac{aa}{aaaaa} = \frac{1}{aaa} = \frac{1}{a^3}.$$

In general, $\dfrac{a^m}{a^n} = \dfrac{aaa \ldots\ldots \text{ to } m \text{ factors}}{aaa \ldots\ldots \text{ to } n \text{ factors}}$

$$= aaa \ldots\ldots \text{ to } m - n \text{ factors (if } m > n);$$

or $$= \frac{1}{aaa \ldots\ldots \text{ to } n - m \text{ factors}} \text{ (if } n > m).$$

Hence, if a power of a number be divided by a *lower* power of the same number, *the quotient is that power of the number of which the exponent is the exponent of the dividend diminished by that of the divisor;* and if any power of a number be divided by a *higher* power of the same number, *the quotient is expressed by* 1 *divided by that power of the number of which the exponent is the exponent of the divisor diminished by that of the dividend.*

The term *power* has so far been restricted to *positive integral powers.*

The above is the **index law** for division.

42. Division of Polynomials by Monomials. The product

$$(a + b + c) \times p = ap + bp + cp. \qquad \S\ 34$$

Therefore, $(ap + bp + cp) \div p = a + b + c.$

But a, b, and c are the quotients obtained by dividing each term, ap, bp, and cp, by p.

Therefore, to divide a polynomial by a monomial, *divide each term of the polynomial by the monomial.*

This is the **distributive law** for division.

43. Division of Polynomials by Polynomials.

If the divisor (one factor) is $a + b + c$,
and the quotient (other factor) is $n + p + q$,

then the dividend (product) is
$$\begin{cases} an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq \end{cases}$$

The first term of the dividend is an, the product of a the first term of the divisor, by n the first term of the quotient. The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n be subtracted from the dividend, ap the first term of the remainder is the product of a, the first term of the divisor, by p, the second term of the quotient. Hence, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. And so on.

Therefore, to divide one polynomial by another:

Divide the first term of the dividend by the first term of the divisor.

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there be a remainder, consider it as a new dividend and proceed as before.

It is of great importance to arrange both dividend and divisor according to the ascending or descending *powers of some common letter*, and to *keep this order* throughout the operation.

(1) Divide

$$22a^2b^2 + 15b^4 + 3a^4 - 10a^3b - 22ab^3 \text{ by } a^2 + 3b^2 - 2ab.$$

$$\begin{array}{rrrrr|l}
3a^4 & -10a^3b & +22a^2b^2 & -22ab^3 & +15b^4 & a^2 - 2ab + 3b^2 \\
3a^4 & -6a^3b & +9a^2b^2 & & & \overline{3a^2 - 4ab + 5b^2} \\
\hline
& -4a^3b & +13a^2b^2 & -22ab^3 & & \\
& -4a^3b & +8a^2b^2 & -12ab^3 & & \\
\hline
& & 5a^2b^2 & -10ab^3 & +15b^4 & \\
& & 5a^2b^2 & -10ab^3 & +15b^4 & \\
\hline
\end{array}$$

The operation of division may be shortened in some cases by the use of parentheses.

(2)
$$\begin{array}{llll|l}
x^3 & +(a+b+c)x^2 & +(ab+ac+bc)x & +abc & x+b \\
x^3 & +(\quad +b \quad)x^2 & & & \overline{x^2+(a+c)x+ac} \\
\hline
& (a \quad +c)x^2 & +(ab+ac+bc)x & & \\
& (a \quad +c)x^2 & +(ab \quad +bc)x & & \\
\hline
& & acx & +abc & \\
& & acx & +abc & \\
\hline
\end{array}$$

44. Detached Coefficients. In Division as in Multiplication, it is convenient to use only the coefficients when the dividend and divisor are expressions involving but one letter, or homogeneous expressions involving but two letters. Thus, the work of Ex. 1 of the last section may be arranged as follows:

$$\begin{array}{rrrrr|l}
3 & -10 & +22 & -22 & +15 & 1-2+3 \\
3 & -6 & +9 & & & \overline{3-4+5} \\
\hline
& -4 & +13 & -22 & & \\
& -4 & +8 & -12 & & \\
\hline
& & 5 & -10 & +15 & \\
& & 5 & -10 & +15 & \\
\hline
\end{array}$$

The quotient is $3a^2 - 4ab + 5b^2$.

45. Special Cases. There are some cases in Division which occur so often in algebraic operations that they should be carefully noticed and remembered.

The student may easily verify the following results:

$$(1)\quad \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

$$(2)\quad \frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4.$$

In general, it will be found that the difference of two like powers of any two numbers is divisible by the difference of the numbers.

$$(3)\quad \frac{a^3+b^3}{a+b}=a^2-ab+b^2.$$

$$(4)\quad \frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$

In general, it will be found that the sum of two like *odd* powers of two numbers is divisible by the sum of the numbers.

Compare the quotients in (3) and (4) with those in (1) and (2).

$$(5)\quad \frac{x^2-y^2}{x-y}=x+y. \qquad (7)\quad \frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3.$$

$$(6)\quad \frac{x^2-y^2}{x+y}=x-y. \qquad (8)\quad \frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3.$$

In general, it will be found that the *difference* of two like *even* powers of two numbers is divisible by the difference and also by the sum of the numbers.

The *sum* of two like *even* powers of two numbers is not divisible by either the sum or the difference of the numbers.

But when the exponent of each of the two like powers

is composed of an *odd* and an *even* factor, the sum of the given powers is divisible by the sum of the powers expressed by the even factor.

Thus, $x^6 + y^6$ is not divisible by $x + y$, or by $x - y$, but is divisible by $x^2 + y^2$.

The quotient may be found as in examples (3) and (4).

It appears, then, that a factor of $x^n - y^n$ can always be found; and that a factor of $x^n + y^n$ can be found *unless n is a power of* 2.

Thus, factors of $x^2 + y^2$, $x^4 + y^4$, $x^8 + y^8$, etc., cannot be found.

Exercise 4.

Divide:

1. $(6a^2b^3c \times 35a^2b^5c^4)$ by $(21a^3b^9c^6 \times 2a^2c^3)$.
2. $39a^3x^2 + 24a^4x^3 + 42a^2x^3 + 27a^4x^2$ by $6a^2x^2$.
3. $35x^3 + 94ax^2 + 52a^2x + 8a^3$ by $5x + 2a$.
4. $x^3 - 5ax^2 - a^2x + 14a^3$ by $x^2 - 3ax - 7a^2$.
5. $81x^4 + 36x^2y^2 + 16y^4$ by $9x^2 - 6xy + 4y^2$.
6. $x^4 + b^4 - a^2x^2 + 2b^2x^2$ by $x^2 + b^2 + ax$.
7. $a^2 - 2b^2 - 3c^2 + ab + 2ac + 7bc$ by $a - b + 3c$.
8. $4x^4 - 5x^2y^2 - 8x^2 - 4y^2 + 4 + y^4$
   by $y^2 + 2x^2 - 2 - 3xy$.
9. $2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2m}$ by $a^n - 2a$.
10. $625x^4 - 81y^4$ by $5x - 3y$.
11. $x^{3n} + y^{3n}$ by $x^n + y^n$.
12. $\dfrac{27a^3}{125} - \dfrac{b^3}{64}$ by $\dfrac{3a}{5} - \dfrac{b}{4}$.
13. $(a + 2b)^3 + (b - 3c)^3$ by $a + 3(b - c)$.
14. $a^m - a^{m+1} + 37a^{m+3} - 55a^{m+4} + 50a^{m+5}$
    by $1 - 3a + 10a^2$.

15. $4h^{x+1} - 30h^{x} + 19h^{x-1} + 5h^{x-2} + 9h^{x-4}$
by $h^{x-3} - 7h^{x-4} + 2h^{x-5} - 3h^{x-6}$.

16. $6x^{m-n+2} + x^{m-n+1} - 22x^{m-n} + 19x^{m-n-1} - 4x^{m-n-2}$
by $3x^{3-n} - 4x^{2-n} + x^{1-n}$.

46. Extension of Meaning. The introduction of negative numbers requires an extension of the meanings of some terms common to arithmetic and algebra. But every such extension of meaning must be consistent with the sense previously attached to the term and with general laws already established.

Addition in algebra does not necessarily imply *augmentation*, as it does in arithmetic. Thus, $7 + (-5) = 2$. The word **sum**, however, is used to denote the result.

Such a result is called the **algebraio sum**, when it is necessary to distinguish it from the *arithmetical sum*, which would be obtained by adding the *absolute values* of the numbers.

The general definition of Addition is, the operation of uniting two or more expressions in a *single expression* written in its simplest form.

The general definition of Subtraction is, the operation of finding from two given expressions, called *minuend* and *subtrahend*, a third expression, called *difference*, which added to the subtrahend will give the minuend.

The general definition of Multiplication is, the operation of finding from two given expressions, called *multiplicand* and *multiplier*, a third expression, called *product*, which may be formed from the multiplicand as the multiplier is formed from unity.

The general definition of Division is, the operation of finding the *other factor* when the *product* of two factors and *one factor* are given.

47. Fundamental Laws. All the operations of algebra are performed subject to the following laws:

I. The commutative law (§§ 21, 31).
II. The associative law (§§ 27, 33).
III. The distributive law (§§ 34, 42).
IV. The index law (§§ 32, 41).

The various meaning of these laws as applied to the four fundamental operations have been explained as they occurred. We shall now simply formulate them, including Subtraction under Addition.

I. The commutative law:

For Addition $a+b=b+a.$

For Multiplication $ab=ba.$

II. The associative law:

For Addition $a+(b+c)=(a+b)+c.$

For Multiplication $abc=a(bc)=(ab)c.$

III. The distributive law:

For Multiplication $n(a+b+c)=na+nb+nc.$

For Division $\dfrac{a+b+c}{n}=\dfrac{a}{n}+\dfrac{b}{n}+\dfrac{c}{n}.$

IV. The index law:

For Multiplication $a^m \times a^n = a^{m+n}.$

For Division $\dfrac{a^m}{a^n}=a^{m-n}$ (if $m>n$);

$\dfrac{a^m}{a^n}=\dfrac{1}{a^{n-m}}$ (if $n>m$).

CHAPTER III.

FACTORS.

In multiplication we determine the *product* of two given factors; it is often important to determine the *factors* of a given product.

48. The simplest case is that in which all the terms of an expression have one common factor. Thus,

(1) $x^2 + xy = x(x + y)$.

(2) $6a^3 + 4a^2 + 8a = 2a(3a^2 + 2a + 4)$.

Frequently the terms of an expression can be arranged so as to show a common factor. Thus,

$$\begin{aligned}(3)\ ac - ad - bc + bd &= (ac - ad) - (bc - bd)\\ &= a(c - d) - b(c - d)\\ &= (a - b)(c - d).\end{aligned}$$

49. The square root of a number is one of the *two equal* factors of that number. Thus, the square root of 25 is 5; for, $25 = 5 \times 5$. The square root of a^4 is a^2; for,

$$a^4 = a^2 \times a^2.$$

In general, the square root of an even power of a number is expressed by writing the number with an exponent equal to one-half the exponent of the power.

50. Since $a^n b^n \times a^n b^n = a^n b^n a^n b^n = a^n a^n b^n b^n = a^{2n} b^{2n}$, $a^n b^n$ is the square root of $a^{2n} b^{2n}$. But a^n is the square root

of a^{2n}, and b^n of b^{2n}. Therefore, the square root of the product of even powers may be found by taking the square root of each factor, and finding the product of the roots.

The square root of a positive number may be either positive or negative; for,

$$a^2 = a \times a,$$

or,

$$a^2 = -a \times -a;$$

but throughout this chapter only the positive value of the square root will be considered.

51. From § 38 it is seen that a trinomial is often the product of two binomials. Conversely, a trinomial may, in certain cases, be resolved into two binomial factors.

(1) To find the factors of $x^2 + 7x + 12$.

The first term of each binomial factor will obviously be x.

The second terms of the two binomial factors must be two numbers of which the *product* is 12, and the *sum* 7.

These two numbers are 4 and 3.

$$\therefore\ x^2 + 7x + 12 = (x + 4)(x + 3).$$

(2) To find the factors of $x^2 - 9x - 36$.

The second terms of the two binomial factors must be two numbers of which the *product* is -36, and the *sum* -9.

These two numbers are -12 and $+3$.

$$\therefore\ x^2 - 9x - 36 = (x - 12)(x + 3).$$

52. Consider trinomials which are perfect squares. These are only particular forms of the trinomials of the last section, but from their importance demand special attention.

(1) To find the factors of $x^2 + 18x + 81$.

The second terms of the two binomial factors must be two numbers of which the *product* is 81, and the *sum* 18.

These two numbers are 9 and 9.

$$\therefore\ x^2 + 18x + 81 = (x + 9)(x + 9) = (x + 9)^2.$$

(2) To find the factors of $x^2 - 18x + 81$.

The second terms of the two binomial factors must be two numbers of which the *product* is 81, and the *sum* -18.

These two numbers are -9 and -9.

$$\therefore\ x^2 - 18x + 81 = (x-9)(x-9) = (x-9)^2.$$

53. An expression in the form of two squares, with the negative sign between them, is the product of two factors which may be determined as follows:

Take the square root of the first number, and the square root of the second number.

The *sum* of these roots will form the first factor;

The *difference* of these roots will form the second factor.

Thus,

(1) $a^2 - b^2 = (a+b)(a-b)$.

(2) $$\begin{aligned}(a-b)^2 - (c-d)^2 &= \{(a-b)+(c-d)\}\{(a-b)-(c-d)\}\\ &= \{a-b+c-d\}\{a-b-c+d\}.\end{aligned}$$

The terms of an expression may often be arranged so as to form two squares with the negative sign between them, and the expression can then be resolved into factors.

(3) $$\begin{aligned}&a^2 + b^2 - c^2 - d^2 + 2ab + 2cd\\ &= a^2 + 2ab + b^2 - c^2 + 2cd - d^2\\ &= (a^2 + 2ab + b^2) - (c^2 - 2cd + d^2)\\ &= (a+b)^2 - (c-d)^2\\ &= \{(a+b)+(c-d)\}\{(a+b)-(c-d)\}\\ &= \{a+b+c-d\}\{a+b-c+d\}.\end{aligned}$$

An expression may often be resolved into three or more factors.

(4) $$\begin{aligned}x^{16} - y^{16} &= (x^8+y^8)(x^8-y^8)\\ &= (x^8+y^8)(x^4+y^4)(x^4-y^4)\\ &= (x^8+y^8)(x^4+y^4)(x^2+y^2)(x^2-y^2)\\ &= (x^8+y^8)(x^4+y^4)(x^2+y^2)(x+y)(x-y).\end{aligned}$$

Any expression of the form $x^n \pm y^n$ may be resolved into factors by the principles of § 45 with one exception; viz., *when the sign is* $+$ *and* n *is a power of* 2.

54. For a trinomial to be a perfect square, the middle term must be *twice the product* of the *square roots* of the first and last terms.

The expression $4x^4 - 37x^2y^2 + 9y^4$ will become a perfect square if $25x^2y^2$ be added to the middle term. We must also subtract $25x^2y^2$ to keep the expression unchanged.

This gives
$$\begin{aligned} &4x^4 - 37x^2y^2 + 9y^4 \\ &= (4x^4 - 12x^2y^2 + 9y^4) - 25x^2y^2 \\ &= (2x^2 - 3y^2)^2 - 25x^2y^2 \\ &= (2x^2 - 3y^2 + 5xy)(2x^2 - 3y^2 - 5xy) \\ &= (2x^2 + 5xy - 3y^2)(2x^2 - 5xy - 3y^2). \end{aligned}$$

55. To find the factors of $6x^2 + x - 12$.

It is evident that the first terms of the two factors may be $6x$ and x, or $2x$ and $3x$, since the product of either of these pairs is $6x^2$.

Likewise, the last terms of the two factors may be 12 and 1, 6 and 2, or 4 and 3 (if we disregard the signs).

From these it is necessary to select such as will produce the middle term of the trinomial. And they are found by trial to be $3x$ and $2x$, and -4 and $+3$.

$$\therefore\ 6x^2 + x - 12 = (3x - 4)(2x + 3).$$

56. The factors, if any exist, of a polynomial of more than three terms can often be found by the application of principles already explained.

Thus, it is seen that the expression

$$x^2 - 2xy + y^2 + 2xz - 2yz + z^2$$

consists of *three* squares and *three* double products, and from § 37, is the square of a *trinomial* which has for terms x, y, z.

It is also seen from the double product $-2xy$, that x and y have *unlike* signs; and from the double product $2xz$, that x and z have *like* signs. Hence,

$$x^2 - 2xy + y^2 + 2xz - 2yz + z^2 = (x - y + z)^2.$$

57. Find the factors of

$$6x^2 - 7xy - 3y^2 - 9x + 30y - 27.$$

The factors of the first three terms are $3x + y$ and $2x - 3y$.

Now -27 must be resolved into two factors such that the sum of the products obtained by multiplying one of these factors by $3x$ and the other by $2x$ shall be $-9x$.

These two factors evidently are -9 and $+3$. Therefore,

$$(6x^2 - 7xy - 3y^2 - 9x + 30y - 27 = (3x + y - 9)(2x - 3y + 3).$$

This result may be verified by actual multiplication.

58. The following method is often convenient for separating a polynomial into its factors:

Find the factors of

$$2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2.$$

(1) Reject the terms that contain z.
(2) Reject the terms that contain y.
(3) Reject the terms that contain x.

Factor the expression that remains in each case.

(1) $2x^2 - 5xy + 2y^2 = (x - 2y)(2x - y)$.
(2) $2x^2 + 7xz + 3z^2 = (x + 3z)(2x + z)$.
(3) $2y^2 - 5yz + 3z^2 = (-2y + 3z)(-y + z)$.

Arrange these three pairs of factors in two rows of three factors each, so that any two factors of each row may have a *common term*. Thus,

$$x - 2y, \quad x + 3z, \quad -2y + 3z;$$
$$2x - y, \quad 2x + z, \quad -y + z.$$

From the first row, select the *terms common to two factors* for one trinomial factor:

$$x - 2y + 3z.$$

From the second row, select the *terms common to two factors* for the other trinomial factor.

$$2x - y + z.$$

Then,

$$2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2 = (x - 2y + 3z)(2x - y + z).$$

When a factor obtained from the first three terms is also a factor of the remaining terms, the expression is easily resolved.

Thus, $x^2 - 3xy + 2y^2 - 3x + 6y = (x - 2y)(x - y) - 3(x - 2y)$
$= (x - 2y)(x - y - 3).$

Exercise 5.

Resolve into factors:

1. $9x^4 + 6x^3 + 3x^2 + 2x.$
2. $2a^4 - 3a^3b - 14a^2 + 21ab.$
3. $5x^3 + 15x^2y - 4xy^2 - 12y^3.$
4. $a^2x^3 - b^2xy^2 - a^2cx^2 + b^2cy^2.$
5. $x^2 + 8x + 7.$
6. $x^2 - 17x + 60.$
7. $x^2 + 7x - 18.$
8. $x^2 - 2x - 24.$
9. $9x^2 + 30x + 25.$
10. $16x^2 - 56x + 49.$
11. $x^2 + x - 72.$
12. $x^2 - 14x - 176.$
13. $81x^4 - 196x^2y^2.$
14. $729a^6 - x^6.$
15. $64x^7 + xy^6.$
16. $(x^2 - y^2)^2 - y^4.$
17. $(a^2 + 2b^2)^2 - a^2b^2.$
18. $(2x - 3y)^2 - (x - 2y)^2.$
19. $(2x^2 - 4x + 7)^2 - x^2(x + 4)^2.$

20. $x^4 - 2(b^2 - c^2)x^2 + b^4 - 2b^2c^2 + c^4.$

21. $15x^2 - 7x - 2.$

22. $11x^2 - 54x + 63.$

23. $21x^2 + 26x - 15.$

24. $70x^2 - 27x - 9.$

25. $x^4 - 2abx^2 - a^4 - a^2b^2 - b^4.$

26. $5x^4 + 4x^3 - 20x - 125.$

27. $2x^4 - 5x^3 - x^2 - 2.$

28. $6x^4 - ax^3 - 2a^2x^2 + 3a^3x - 2a^4.$

29. $12x^5 + 10x^4y - 12x^3y^2 - 6x^2y^3 - 4y^5.$

HIGHEST COMMON FACTOR.

59. A **common factor** of two or more expressions is an expression which is contained in each of them without a remainder.

Two expressions which have no common factor except 1, are said to be *prime* to each other.

The **highest common factor** of two or more expressions is the product of all the factors common to the expressions.

For brevity, H. C. F. will be used for highest common factor.

Ex. Find the H. C. F. of

$$8a^2x^2 - 24a^2x + 16a^2 \text{ and } 12ax^2y - 12axy - 24ay.$$

$$\begin{aligned} 8a^2x^2 - 24a^2x + 16a^2 &= 8a^2(x^2 - 3x + 2) \\ &= 2^3a^2(x-1)(x-2); \\ 12ax^2y - 12axy - 24ay &= 12ay(x^2 - x - 2) \\ &= 2^2 \times 3ay(x+1)(x-2). \end{aligned}$$

$$\therefore \text{ the H.C.F.} = 2^2a(x-2) = 4a(x-2).$$

Hence, to find the H. C. F. of two or more expressions:

Resolve each expression into its lowest factors.

Select from these the lowest power of each common factor, and find the product of these powers.

60. When it is required to find the H. C. F. of two or more expressions which cannot readily be resolved into their factors, the method to be employed is similar to that of the corresponding case in arithmetic. And as that method consists in obtaining pairs of continually decreasing numbers which contain as a factor the H. C. F. required; so in algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H. C. F. required.

The method depends upon two principles:

I. *Any factor of an expression is a factor also of any multiple of that expression.*

Thus, if F represent a factor of an expression A, so that $A = nF$, then $mA = mnF$. That is, mA contains the factor F.

II. *Any common factor of two expressions is a factor of the sum or difference of any multiples of the expressions.*

Thus, if F represent a common factor of the expressions A and B so that

$$A = mF, \text{ and } B = nF;$$

then $$pA = pmF, \text{ and } qB = qnF.$$

Hence, $$pA \pm qB = pmF \pm qnF,$$
$$= (pm \pm qn)F.$$

That is, $pA \pm qB$ contains the factor F.

61. The general proof of this method as applied to *numbers* is as follows:

Let a and b be two numbers, of which a is the greater. The operation may be represented by:

$b)a(p$	$42)154(3$	$nF)mF(p$
$\underline{pb}$	$\underline{126}$	$\underline{pnF}$
$c)b(q$	$28)42(1$	$cF)nF(q$
$\underline{qc}$	$\underline{28}$	$\underline{qcF}$
$d)c(r$	$14)28(2$	$F)cF(c$
$\underline{rd}$	$\underline{28}$	$\underline{cF}$

p, q, and r represent the several quotients,
c and d represent the remainders,
and d is supposed to be contained exactly in c.

The numbers represented are all integral.

Then
$$\begin{aligned} c &= rd, \\ b &= qc + d = qrd + d = (qr+1)d, \\ a &= pb + c = pqrd + pd + rd \\ &= (pqr + p + r)d. \end{aligned}$$

$\therefore$ d is *a common factor* of a and b.

It remains to show that d is the *highest* common factor of a and b.

Let f represent the highest common factor of a and b.

Now $c = a - pb$, and f is a common factor of a and b.

$\therefore$ by (II.) f is a factor of c.

Also, $d = b - qc$, and f is a common factor of b and c.

$\therefore$ by (II.) f is a factor of d.

That is, *d contains the highest common factor* of a and b.

But it has been shown that d is a common factor of a and b.

$\therefore$ d is the *highest common factor* of a and b.

NOTE. The second operation represents the application of the method to a particular case. The third operation is intended to represent clearly that every remainder in the course of the operation contains as a factor the H. C. F. sought, and that this is the *highest factor common* to that remainder and the preceding divisor.

62. This method is only needed to determine the *compound* factor of the H. C. F. *Simple* factors of the given expressions should be separated, and the highest common factor of these factors reserved to be multiplied into the compound factor obtained.

Modifications of this method are sometimes needed.

(1) Find the H. C. F. of

$$4x^2-8x-5 \text{ and } 12x^2-4x-65.$$

$$\begin{array}{r} 4x^2-8x-5)\,12x^2-4x-65\,(3 \\ \underline{12x^2-24x-15} \\ 20x-50 \end{array}$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x-50$ be made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

Now, it is to be remembered that the H. C. F. sought is *contained in the remainder* $20x-50$, and that it is a *compound factor*. Hence if the *simple factor* 10 be removed, the H. C. F. must still be contained in $2x-5$, and therefore the process may be continued with $2x-5$ for a divisor.

$$\begin{array}{r} 2x-5)\,4x^2-8x-5\,(2x+1 \\ \underline{4x^2-10x} \\ 2x-5 \\ \underline{2x-5} \end{array}$$

$\therefore$ the H. C. F. $=2x-5$.

(2) Find the H. C. F. of

$$21x^3-4x^2-15x-2 \text{ and } 21x^3-32x^2-54x-7.$$

Writing only the coefficients (§ 44), the work is as follows:

$$\begin{array}{r} 21-4-15-2)\,21-32-54-7\,(1 \\ \underline{21-4-15-2} \\ -28-39-5 \end{array}$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, the expression $21x^3-4x^2-15x-2$ must be *multiplied* by the simple factor 4 to make its first term divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H.C.F. sought; for the H.C.F. contains only factors *common to the remainder and the last divisor*, and 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$\begin{array}{lrl}
 & 28+39+5)\,84-16-60-8\,(3 & \\
 & \underline{84+117+15} & \\
 & -133-75-8 & \\
\text{Multiply by } -4, & \underline{-4} & \\
 & 532+300+32\,(19 & \\
 & \underline{532+741+95} & \\
\text{Divide by } -63, & -63)\,\underline{-441-63} & \\
 & 7+1 &
\end{array}$$

$$\begin{array}{lr}
 & 7+1)\,28+39+5\,(4+5 \\
 & \underline{28+4} \\
 & 35+5 \\
\therefore \text{ the H.C.F. is } 7x+1. & \underline{35+5}
\end{array}$$

In practice the work is most conveniently arranged as follows:

$21-4-15-2$	$21-32-54-7$	1
4	$21-4-15-2$	
$84-16-60-8$	$-1)-28-39-5$	
$84+117+15$	$28+39+5$	$3+19$
$-133-75-8$	$28+4$	
-4	$35+5$	
$532+300+32$	$35+5$	
$532+741+95$		
$-63)-441-63$		
$7+1$		$4+5$

$\therefore$ the H.C.F. is $7x+1$.

In the preceding work each quotient is placed opposite the corresponding divisor; but the position of the quotients is evidently a matter of indifference.

63. From the foregoing examples it will be seen that, in the algebraic process of finding the highest common factor, the following steps, in the order here given, must be carefully observed:

I. Simple factors of the given expressions are to be removed from them, and the highest common factor of these is to be reserved as a factor of the H. C. F. sought.

II. The resulting compound expressions are to be arranged according to the *descending* powers of a common letter; and that expression which is of the lower degree is to be taken for the divisor; or, if both are of the same degree, that whose first term has the smaller coefficient.

III. Each division is to be continued until the remainder is of lower degree than the divisor.

IV. If the final remainder of any division is found to contain a factor that is not a *common* factor of the given expressions, *this factor is to be removed;* and the resulting expression is to be used as the next divisor.

V. A dividend whose first term is not exactly divisible by the first term of the divisor, is to be *multiplied* by such an expression as will make it thus divisible.

The H. C. F. of three expressions will be obtained by finding the H. C. F. of two of them, and then of that and the third expression.

LOWEST COMMON MULTIPLE.

64. A **common multiple** of two or more expressions is an expression which is exactly divisible by each of them.

The **lowest common multiple** of two or more expressions

is the product of all the factors of the expressions, each factor being written with its highest exponent.

The lowest common multiple of two expressions which have no common factor will be their product.

For brevity L. C. M. will be used for lowest common multiple.

Find the L. C. M. of $12a^2c$, $14bc^2$, $36ab^2$.

$$12a^2c = 2^2 \times 3a^2c,$$

$$14bc^2 = 2 \times 7bc^2,$$

$$36ab^2 = 2^2 \times 3^2ab^2.$$

$$\therefore \text{ the L. C. M.} = 2^2 \times 3^2 \times 7a^2b^2c^2 = 252a^2b^2c^2.$$

65. When the expressions cannot be readily resolved into their factors, the expressions may be resolved by finding their H. C. F.

Find the L. C. M. of

$$6x^3 - 11x^2y + 2y^3 \text{ and } 9x^3 - 22xy^2 - 8y^3.$$

$$\begin{array}{r|r|l}
6 - 11 + 0 + 2 & 9 + 0 - 22 - 8 & 3 \\
6 - 8 - 4 & 2 \qquad\qquad\quad & \\
\hline
-3 + 4 + 2 & 18 + 0 - 44 - 16 & \\
-3 + 4 + 2 & 18 - 33 + 0 + 6 & \\
\hline
 & 11\underline{)33 - 44 - 22} & \\
 & 3 - 4 - 2 & 2 - 1
\end{array}$$

Hence, $6x^3 - 11x^2y + 2y^3 = (2x - y)\ (3x^2 - 4xy - 2y^2)$,

and $9x^3 - 22xy^2 - 8y^3 = (3x + 4y)(3x^2 - 4xy - 2y^2)$.

$\therefore$ the L. C. M. $= (2x - y)(3x + 4y)(3x^2 - 4xy - 2y^2)$.

In this example we find the H. C. F. of the given expression, and divide each of them by the H. C. F.

Instead of dividing both expressions by their H. C. F., we might have divided only one expression, and have multiplied the quotient by the other expression.

The object of finding the H. C. F. is to obtain some means of *factoring* the given expressions.

Exercise 6.

Find the H. C. F. of:

1. $12x^2-17x+6,\ 9x^2+6x-8.$
2. $x^4-a^4,\ x^2+3ax-4a^2,\ x^2-5ax+4a^2.$
3. $x^4-6x^3+13x^2-12x+4,\ x^4-4x^3+8x^2-16x+16.$
4. $3x^4-x^3-2x^2+2x-8,\ 6x^4+13x^3+3x^2+20x.$
5. $96x^4+8x^3-2x,\ 32x^3-24x^2-8x+3.$
6. $x^4+5x^3-7x^2-9x-10,\ 2x^4-4x^3+4x-8.$
7. $2x^3-16x+6,\ 5x^6+15x^5+5x+15.$
8. $2a^4+3a^3x-9a^2x^2,\ 6a^4x-3ax^4-17a^3x^2+14a^2x^3.$
9. $2a^5-4a^4+8a^3-12a^2+6a,$
   $3a^6-3a^5-6a^4+9a^3-3a^2.$
10. $3x^3-7x^2y-y^3+5xy^2,\ x^2y+3xy^2-3x^3-y^3,$
    $3x^3+5x^2y+xy^2-y^3.$
11. $36x^7-28x^5+32x^4+8x^3-16x^2,$
    $12x^5-14x^4-20x^3+10x^2+4x.$

Find the L. C. M. of:

12. $x^2-3x-4,\ x^2-x-12,\ x^2+5x+4.$
13. $6x^2-13x+6,\ 6x^2+5x-6,\ 9x^2-4.$
14. $3x^4-x^3-2x^2+2x-8,\ 6x^3+13x^2+3x+20.$
15. $15a^3x^4+10a^4x^3+4a^5x^2+6a^6x-3a^7,$
    $12x^4+38ax^3+16a^2x^2-10a^3x.$
16. $2x^4+x^3-8x^2-x+6,\ 4x^4+12x^3-x^2-27x-18,$
    $4x^4+4x^3-17x^2-9x+18.$

CHAPTER IV.

FRACTIONS.

66. An algebraic expression is **integral** when it consists of a number of terms connected by $+$ and $-$ signs, each term being the product of a coefficient into powers of the letters involved.

In an integral algebraic expression the *coefficients* may be fractional. Thus, $x^3 - \frac{4}{5}ax^2 + \frac{2}{3}a$ is an *integral* algebraic expression.

67. An **algebraic fraction** is the quotient of two integral expressions, and is generally written in the form $\frac{a}{b}$.

The dividend, a, is called the **numerator**; the divisor, b, the **denominator.**

The numerator and denominator are called the **terms** of the fraction.

68. Since the quotient is unchanged if the dividend and divisor are both multiplied (or divided) by the same factor, the value of a fraction is unchanged if the numerator and denominator are multiplied (or divided) by the same factor.

69. To reduce a fraction to lower terms,

Divide the numerator and denominator by any common factor.

A fraction is expressed in its **lowest terms** when both numerator and denominator are divided by their H. C. F.

(1) Reduce to lowest terms $\dfrac{6x^2-5x-6}{8x^2-2x-15}$.

By § 55, $\dfrac{6x^2-5x-6}{8x^2-2x-15}=\dfrac{(2x-3)(3x+2)}{(2x-3)(4x+5)}=\dfrac{3x+2}{4x+5}$.

(2) Reduce to lowest terms $\dfrac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}$.

Since no common factor can be determined by inspection, it is necessary to find the H. C. F. of the numerator and denominator by the method of division.

We find the H. C. F. to be $a-2$.

Now, if $a^3-7a^2+16a-12$ be divided by $a-2$, the result is a^2-5a+6; and if $3a^3-14a^2+16a$ be divided by $a-2$, the result is $3a^2-8a$.

$$\therefore \frac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}=\frac{a^2-5a+6}{3a^2-8a}.$$

70. Mixed Expressions. If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to the form of a mixed or integral expression *by dividing the numerator by the denominator.*

The quotient will be the integral expression; the remainder (if any) will be the numerator, and the divisor the denominator, of the fractional expression.

To reduce a mixed expression to a fractional form,

Multiply the integral expression by the denominator, to the product annex the numerator, and under the result write the denominator.

The dividing line has the force of a vinculum or parenthesis affecting the numerator; therefore, if a *minus sign* precede the dividing line, and this line be removed, the sign of *every term* of the numerator must be *changed.*

Thus, $n-\dfrac{a-b}{c}=\dfrac{cn-(a-b)}{c}=\dfrac{cn-a+b}{c}$.

71. Lowest Common Denominator. To reduce fractions to equivalent fractions having the lowest common denominator.

Find the L. C. M. *of the denominators.*

Divide the L. C. M. *by the denominator of each fraction.*

Multiply the first numerator by the first quotient, the second numerator by the second quotient, and so on.

The products will be the numerators of the equivalent fractions.

The L. C. M. *of the given denominators will be the denominator of each of the equivalent fractions.*

Thus, $\frac{3x}{4a^2}, \frac{2y}{3a}, \frac{5}{6a^3}$

are equal to $\frac{9ax}{12a^3}, \frac{8a^2y}{12a^3}, \frac{10}{12a^3}$, respectively.

The multipliers $3a$, $4a^2$, and 2, being obtained by dividing $12a^3$, the L. C. M. of the denominators, by the respective denominators of the given fractions.

72. Addition and Subtraction of Fractions. To add fractions:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Add the numerators of the equivalent fractions.

Write the result over the lowest common denominator.

To subtract one fraction from another we proceed as in addition, except that we *subtract* the numerator of the subtrahend from that of the minuend.

(1) Simplify $\frac{3a-4b}{7} - \frac{2a-b+c}{3} + \frac{13a-4c}{12}$.

The L. C. D. is 84.

The multipliers are 12, 28, and 7 respectively.

$$\begin{array}{rl} 36a - 48b \phantom{{}-28c} & = \text{1st numerator,} \\ -56a + 28b - 28c & = \text{2d numerator,} \\ 91a \phantom{{}+28b} - 28c & = \text{3d numerator.} \\ \hline 71a - 20b - 56c & = \text{sum of numerators.} \end{array}$$

$$\therefore \frac{3a-4b}{7}-\frac{2a-b+c}{3}+\frac{13a-4c}{12}=\frac{71a-20b-56c}{84}.$$

Since the *minus sign* precedes the second fraction, the signs of all the terms of the numerator of this fraction are changed after being multiplied by 28.

(2) Simplify $\dfrac{y^2}{x^2-y^2}-\dfrac{x-y}{x+y}+1+\dfrac{2xy}{x^2+y^2}.$

The L. C. D. is $(x+y)(x-y)(x^2+y^2)$.

The multipliers are x^2+y^2, $(x-y)(x^2+y^2)$, $(x+y)(x-y)$, (x^2+y^2), $(x+y)(x-y)$, respectively.

$$\begin{array}{rl}
x^2y^2 \qquad\qquad + y^4 &= \text{1st numerator,}\\
-x^4+2x^3y-2x^2y^2+2xy^3-y^4 &= \text{2d numerator,}\\
x^4 \qquad\qquad\qquad\qquad -y^4 &= \text{3d numerator,}\\
2x^3y \qquad\qquad -2xy^3 \quad &= \text{4th numerator.}\\
\hline
4x^3y-\ x^2y^2 \qquad\qquad -y^4 &= \text{sum of numerators.}
\end{array}$$

$$\therefore \text{Sum of fractions} = \frac{4x^3y-x^2y^2-y^4}{x^4-y^4}.$$

73. Since $\dfrac{ab}{b}=a$, and $\dfrac{-ab}{-b}=a$,

it is evident that if the signs of both numerator and denominator be changed, the value of the fraction is not altered.

Since changing the sign before the fraction is equivalent to changing the sign before every term of the numerator or denominator, therefore *the sign before every term of the numerator or denominator may be changed, provided the sign before the fraction is changed.*

Since, also, the product of $+a$ multiplied by $+b$ is ab, and the product of $-a$ multiplied by $-b$ is ab, the signs of *two factors*, or of *any even number of factors*, of the numerator or denominator of a fraction may be changed without altering the value of the fraction.

By the application of these principles, fractions may often be changed to a form more convenient for addition or subtraction.

Ex. Simplify $\dfrac{2}{x}-\dfrac{3}{2x-1}+\dfrac{2x-3}{1-4x^2}.$

Change the signs before the terms of the denominator of the third fraction, and change the sign before the fraction.

The result is,

$$\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1},$$

in which the several denominators are written in similar form.

The L. C. D. is $x(2x-1)(2x+1)$.

$$\begin{array}{rl}
8x^2 \qquad\quad -2 & = \text{1st numerator,} \\
-6x^2-3x \qquad & = \text{2d numerator,} \\
\underline{-2x^2+3x \qquad} & = \text{3d numerator.} \\
-2 & = \text{sum of numerators.}
\end{array}$$

$$\therefore \text{ Sum of the fractions} = \frac{-2}{x(2x-1)(2x+1)}.$$

74. Multiplication of Fractions. Let it be required to find the product of the two fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$.

If we multiply the dividend a by c, we multiply the quotient $\dfrac{a}{b}$ by c; if we multiply the divisor b by d, we divide the quotient $\dfrac{a}{b}$ by d. Hence, the product of the two fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$ is $\dfrac{ac}{bd}$. Therefore, to find the product of two fractions,

Find the product of the numerators for the numerator of the product, and the product of the denominators for the denominator of the product.

75. Division of Fractions. Multiplying by the reciprocal of a number is equivalent to dividing by the number. Thus, multiplying by $\frac{1}{4}$ is equivalent to dividing by 4.

The reciprocal of a fraction is the fraction with its terms interchanged. Therefore, to divide by a fraction,

Interchange the terms of the fraction and multiply by the resulting fraction.

If the divisor be an integral expression, it may be changed to the fractional form.

76. A **complex fraction** is one which has a fraction in the numerator, or in the denominator, or in both.

To simplify a complex fraction,

Divide the numerator by the denominator.

It is often shorter to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator.

Exercise 7.

Reduce to lowest terms:

1. $\dfrac{42a^3-30a^2x}{35ax^2-25x^3}$.

2. $\dfrac{2x^3+5x^2-12x}{7x^3+25x^2-12x}$.

3. $\dfrac{6a^2c^2-2a^4+18c^2-6a^2}{4a^4+2a^2c^2+12a^2+6c^2}$.

4. $\dfrac{x^4+(2b^2-a^2)x^2+b^4}{x^4+2ax^3+a^2x^2-b^4}$.

5. $\dfrac{6x^5-9x^4+11x^3+6x^2-10x}{4x^6+10x^5+10x^4+4x^3+60x^2}$.

Simplify:

6. $\dfrac{3x-2y}{3}-\dfrac{4y+2x}{5}+\dfrac{22y-9x}{15}$.

7. $\dfrac{2}{3a}-\dfrac{1}{2b}-\dfrac{2a+3}{6a^2}+\dfrac{1}{2x^2}+\dfrac{3a-2b}{6ab}$.

8. $\dfrac{3}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}.$

9. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}-\dfrac{a-c}{(a-b)(b-c)}.$

10. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}.$

11. $\dfrac{6x^2-17x+12}{12x^2-25x+12}+\dfrac{27x^2+18x-24}{12x^2+7x-12}+\dfrac{25x^2-25x+6}{20x^2-23x+6}.$

12. $\dfrac{2a^8x^7}{3b^8}\times\dfrac{5a^4b^5}{4c^4x^6}\times\dfrac{15b^2c^3}{4a^9x}\div\dfrac{25a^4x}{18ab^2c^3}.$

13. $\left(\dfrac{x^4-y^4}{x^2-y^2}\div\dfrac{x+y}{x^2-xy}\right)\div\left(\dfrac{x^2+y^2}{x-y}\div\dfrac{x+y}{xy-y^2}\right).$

14. $\left(\dfrac{a^2+b^2}{b}-a\right)\left(\dfrac{a^2-b^2}{a^3+b^3}\right)\div\left(\dfrac{1}{b}-\dfrac{1}{a}\right).$

15. $\dfrac{x^2-7x+12}{x^2+5x+6}\times\dfrac{x^2+x-2}{x^2-5x+4}\times\dfrac{2x^2+5x-3}{3x^2-7x-6}.$

16. $\dfrac{6a^2-a-2}{8a^2-2a-3}\times\dfrac{8a^2-10a+3}{12a^2+a-6}\times\dfrac{12a^2+17a+6}{6a^2+a-2}.$

17. $\dfrac{\dfrac{2x+y}{y}-\dfrac{y}{2x+y}}{\dfrac{x}{x+y}-\dfrac{x+y}{x}}.$

18. $\dfrac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}.$

19. $\dfrac{\left(a^2+\dfrac{b^4}{a^2-b^2}\right)(a^2+b^2)}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}.$

20. $\dfrac{\dfrac{64a^3-96a^2x+36ax^2}{36a^2-729x^2}}{\dfrac{48a^2-27x^2}{8a^2-72ax+162x^2}}.$

CHAPTER V.

SIMPLE EQUATIONS.

77. Two different expressions which involve the same symbols will generally have different values for assumed values of the several symbols; for certain values of the symbols involved the two expressions may have the *same* value.

78. An **equation** is a statement that two expressions have the same value; that is, a statement that two expressions represent the same number.

Every equation consists of two expressions connected by the sign of equality; the two expressions are called the **sides** or **members** of the equation.

An equation will in general not hold true for all values of the symbols involved; it will hold true for only those values which give to the two members the same value.

Thus, the equation,

$$4x^2 - 3x + 5 = 3x^2 + 4x - 5,$$

holds true when for x we put 2, since both members then have the value 15; also when for x we put 5, since both members then have the value 90. If we give to x any other value, the two members will be found to have different values, and the equation will not hold true.

79. An **equation of condition** is an equation which holds true for only *certain particular values* of the symbols involved.

An **identical equation**, or an **identity**, is an equation which holds true for *all values* of the symbols involved.

The two members of an identical equation are **identical expressions.**

In identical equations it is customary to use the sign $\equiv$, called the **sign of identity**, instead of the sign of equality.

Thus, the two expressions $(x+y)^2$ and $x^2+2xy+y^2$ have the same value for all values of x and y, and we accordingly write the identity,

$$(x+y)^2 \equiv x^2+2xy+y^2.$$

This is read, $(x+y)^2$ is identically equal to $x^2+2xy+y^2$;
or, $(x+y)^2$ is identical with $x^2+2xy+y^2$.

Wherever the term *equation* is used, it is to be understood that an equation of condition is meant, unless the contrary is expressly stated.

80. In any particular problem we have two kinds of numbers to consider:

(1) Numbers which are either given, or supposed to be given, in the problem under consideration. Such numbers are called **known numbers**; if given, they are generally represented by figures; if only supposed to be given, by the first letters of the alphabet.

(2) Numbers which are not given in the problem under consideration, but are to be found from certain given relations to the given numbers. Such numbers are called **unknown numbers,** and are generally represented by the last letters of the alphabet.

The relations between the known and unknown numbers are generally expressed by means of equations.

To be able to determine *all* the unknown numbers, we must have as many equations as there are unknown numbers. If there are two or more equations, we have a system of **simultaneous equations.**

81. To **solve** an equation, or a system of simultaneous equations, is to find the unknown numbers involved.

82. The **degree** of an equation is the same as the sum, in the term in which that sum is greatest, of the exponents of the several unknown numbers involved in the equation.

If the equation involves but one unknown number, the degree is the same as the exponent of the highest power of the unknown number involved in the equation.

Equations of the first, second, third, and fourth degrees are called, respectively, *simple equations*, *quadratic equations*, *cubic equations*, and *biquadratic equations*.

83. **Literal equations** are equations in which some or all of the given numbers are represented by letters.

84. An equation which involves but one unknown number, represented for example by x, will hold true for those values of x which give to the two members the same value (§ 78), and for no other values of x. The values of x for which the equation holds true are called the **roots** of the equation.

Thus, the roots of the equation $4x^2 - 3x + 5 = 3x^2 + 4x - 5$ are 2 and 5.

To **solve** an equation which involves one unknown number is therefore to find its roots.

85. The various methods of solving equations are based mainly upon the following general principle:

If similar operations be performed upon equal numbers, the results will be equal numbers.

Thus, the two members of a given equation are equal numbers. If the two members be increased by, diminished by, multiplied by, or divided by, equal numbers, the results will be equal numbers. Similarly, if the two members be raised to like powers, or if like roots of the two members be taken, the results will be equal numbers (§ 18).

86. *Any term may be transposed from one side of an equation to the other, provided its sign be changed.*

Suppose we have	$x + a = b.$
Now,	$a = a.$
Subtract,	$x \quad = b - a.$

The a which appeared in the left member with the positive sign, now appears in the right member with the negative sign. Similarly for any other equation.

87. The signs of all the terms on each side of an equation may be changed; for this is in effect transposing every term.

88. To solve an equation with one unknown number,

Transpose all the terms involving the unknown number to the left side, and all the other terms to the right side: combine the like terms, and divide both sides by the coefficient of the unknown number.

To *verify* the result, substitute the value of the unknown number in the original equation.

Ex. Solve $(x-2)(x+4) = (x+1)(x+2)$.

Multiply out, $x^2 + 2x - 8 = x^2 + 3x + 2,$

or $2x - 8 = 3x + 2.$

Transpose, $2x - 3x = 2 + 8,$

$$-x = 10,$$

$$x = -10. \text{ Ans.}$$

89. Fractional Equations. To clear an equation of fraction,

Multiply each term by the L. C. M. *of the denominators.*

If a fraction is preceded by a *minus sign*, the sign of every term of the numerator must be *changed* when the denominator is removed (§ 73).

(1) $\frac{x}{3}-\frac{x-1}{11}=x-9.$

Multiply by 33, the L. C. M. of the denominators.

Then, $11x-3x+3=33x-297.$

Transpose and combine, $-25x=-300.$

$\therefore x=12.$

Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x, the first term of the numerator, is changed to $-$; and the $-$ before 1, the second term of the numerator, is changed to +.

If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn.

(2) $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}.$

Multiply both sides by 14.

Then, $8x+5+\frac{49x-21}{3x+1}=8x+12.$

Transpose and combine, $\frac{49x-21}{3x+1}=7.$

Multiply by $3x+1$, $49x-21=21x+7.$

$\therefore x=1.$

Exercise 8.

Solve:

1. $8(10-x)=5(x+3).$
2. $2x-3(2x-3)=1-4(x-2).$
3. $(x-5)(x+6)=(x-1)(x-2).$
4. $(2x+3)(3x-2)=x^2+x(5x+3).$
5. $(x-3)(x+5)=(x+1)(2x-3)-x^2.$

6. $(x+4)(x-2)=(x+3)(3x+4)-(2x+1)(x-6).$

7. $(x-3)(2x+5)=x(x+4)+(x+1)(x+3).$

8. $(x+2)^2+3x=(x-2)^2+5(16-x).$

9. $(x-3)^2+(x+4)^2=(x-2)^2+(x+3)^2.$

10. $\frac{3x}{5}-\frac{x}{6}=\frac{26}{15}.$

11. $\frac{x-2}{3x-5}=\frac{6}{19}.$

12. $\frac{3x-5}{2x+10}=\frac{2}{3}.$

13. $\frac{3(5x-3)}{2(4x+3)}=\frac{6}{5}.$

14. $\frac{5x-6}{5}-\frac{3x}{4}=\frac{x-9}{10}.$

15. $\frac{12-3x}{4}-\frac{3x-11}{3}=1.$

16. $\frac{4x+17}{x+3}+\frac{3x-10}{x-4}=7.$

17. $\frac{x-3}{2x+1}+\frac{2x-1}{4x-3}=1.$

18. $\frac{4x+3}{3x+4}-\frac{3x-4}{4x-3}=\frac{7}{12}.$

19. $\frac{6x+7}{3}-\frac{3}{x+2}=2x+\frac{1}{2}.$

20. $\frac{2x+1}{a+1}+\frac{2x}{a}=5.$

21. $\frac{ax-b}{c}-\frac{bx+c}{a}=abc.$

22. $\frac{x+a}{3(x+b)}+\frac{x+b}{2(x+a)}=\frac{5}{6}.$

23. $\frac{x-2a}{x+3a}-\frac{13a^2-2x^2}{x^2-9a^2}=3.$

24. $\frac{a}{x}+\frac{x}{a}+\frac{a(x-a)}{x(x+a)}-\frac{x(x+a)}{a(x-a)}=\frac{ax}{a^2-x^2}-2.$

90. Problems. In the statement of problems it is to be remembered that the *unit* of the quantity sought is always given, and it is only the *number* of such units that is to be found. We have nothing to do with the quantities themselves; it is only *numbers* with which we have to deal. Thus, x must never be put for a distance, time, weight, etc., but for a *number* of miles, days, pounds, etc.

(1) A and B had equal sums of money; B gave A \$5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

Let $x =$ *number* of dollars each had.

Then $x + 5 =$ *number* of dollars A had after receiving \$5,

and $x - 5 =$ *number* of dollars B had after giving A \$5.

$$\therefore\ 3(x+5) = 11(x-5),$$
$$3x + 15 = 11x - 55,$$
$$-8x = -70,$$
$$x = 8\tfrac{3}{4}.$$

Therefore each had \$8.75.

(2) A can do a piece of work in 5 days, and B can do it in 4 days. How long will it take A and B to do the work together?

Let $x =$ the number of days it will take A and B together.

Then $\frac{1}{x} =$ the part they can do together in one day.

Now, $\frac{1}{5} =$ the part A can do in one day,

and $\frac{1}{4} =$ the part B can do in one day,

$\therefore\ \frac{1}{5} + \frac{1}{4} =$ the part A and B can do together in one day.

$$\therefore\ \frac{1}{5} + \frac{1}{4} = \frac{1}{x},$$
$$4x + 5x = 20,$$
$$9x = 20,$$
$$x = 2\tfrac{2}{9}.$$

Therefore they can do the work together in $2\frac{2}{9}$ days.

Exercise 9.

1. The difference of two numbers is 3; and three times the greater number exceeds twice the less by 18. Find the numbers.

2. If a certain number be increased by 16, the result is seven times the third part of the number. Find the given number.

3. A boy was asked how many marbles he had. He replied, "If you take away 8 from twice the number I have, and divide the remainder by 3, the result is just one-half the number." How many marbles had he?

4. The sum of the denominator and twice the numerator of a certain fraction is 26. If 3 be added to both numerator and denominator, the resulting fraction is $\frac{2}{3}$. Find the given fraction.

5. A courier sent away with a despatch travels uniformly at the rate of 12 miles per hour; 2 hours after his departure a second courier starts to overtake the first, travelling uniformly at the rate of $13\frac{1}{2}$ miles per hour. In how many hours will the second courier overtake the first?

6. Solve the above problem when the respective rates of the first and second couriers are a and b miles per hour, and the interval between their departures is c hours.

7. A certain railroad train travels at a uniform rate. If the rate were 6 miles per hour faster, the distance travelled in 8 hours would exceed by 50 miles the distance travelled in 11 hours at a rate 7 miles per hour less than the actual rate. Find the actual rate of the train.

8. A can do a piece of work in 10 days; A and B together can do it in 7 days. In how many days can B do it alone?

9. A can do a piece of work in a days; A and B together can do it in b days. In how many days can B do it alone?

10. If A can do a piece of work in $2m$ days, B and A together in n days, and A and C in $m+\frac{n}{2}$ days, how long will it take them to do the work together?

11. A boatman moves 5 miles in $\frac{3}{4}$ of an hour, rowing with the tide; to return it takes him $1\frac{1}{2}$ hours, rowing against a tide one-half as strong. What is the velocity of the stronger tide?

12. A boatman, rowing with the tide, moves a miles in b hours. Returning, it takes him c hours to accomplish the same distance, rowing against a tide m times as strong as the first. What is the velocity of the stronger tide?

13. If A, who is travelling, makes $\frac{1}{2}$ of a mile more per hour, he will be on the road only $\frac{4}{5}$ of the time; but if he makes $\frac{1}{2}$ of a mile less per hour, he will be on the road $2\frac{1}{2}$ hours more. Find the distance and the rate.

14. The circumference of a fore wheel of a carriage is a feet; that of a hind wheel, b feet. What distance will the carriage have passed over, when a fore wheel has made n more revolutions than a hind wheel?

15. A wine merchant has two kinds of wine which he sells, one at a dollars, and the other at b dollars per gallon. He wishes to make a mixture of l gallons, which shall cost him on the average m dollars a gallon. How many gallons must he take of each?

Discuss the question (i.) when $a=b$; (ii.) when a or $b=m$; (iii.) when $a=b=m$; (iv.) when $a>b$ and $<m$; (v.) when $a>b$ and $b>m$.

CHAPTER VI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

91. Equations that express *different* relations between the unknown numbers are called **independent equations.**

Thus, $x + y = 10$ and $x - y = 2$ are independent equations; they express *different* relations between x and y. But $x + y = 10$ and $3x + 3y = 30$ are not independent equations; one is derived immediately from the other, and both express the *same* relation between the unknown numbers.

92. Equations that are satisfied by the *same values* of the unknown numbers are called **simultaneous equations.**

93. Simultaneous equations are solved by combining the equations so as to obtain a single equation with one unknown number; this process is called **elimination.**

There are three methods of elimination in general use:

I. By Addition or Subtraction.

II. By Substitution.

III. By Comparison.

We shall give one example of each method.

(1) Solve:
$$\left.\begin{aligned} 2x - 3y &= 4 \\ 3x + 2y &= 32 \end{aligned}\right\} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 2 and (2) by 3,
$$\begin{aligned} 4x - 6y &= 8 \qquad (3) \\ 9x + 6y &= 96 \qquad (4) \end{aligned}$$

Add (3) and (4), $13x = 104$

$\therefore\ x = 8.$

Substitute the value of x in (2), $24 + 2y = 32.$

$\therefore\ y = 4.$

In this solution y is eliminated by *addition*.

(2) Solve:
$$\left.\begin{aligned}2x+3y&=8\\3x+7y&=7\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose $3y$ in (1), $2x = 8 - 3y$.

Divide by coefficient of x, $x = \dfrac{8-3y}{2}$. (4)

Substitute the value of x in (2),

$$3\left(\frac{8-3y}{2}\right) + 7y = 7.$$

$$\frac{24-9y}{2} + 7y = 7.$$

$$24 - 9y + 14y = 14.$$

$$5y = -10.$$

$$\therefore y = -2.$$

Substitute the value of y in (4), $\therefore x = 7$.

In this solution y is eliminated by *substitution*.

(3) Solve:
$$\left.\begin{aligned}2x-9y&=11\\3x-4y&=7\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\end{aligned}$$

Transpose $9y$ in (1) and $4y$ in (2),

$$2x = 11 + 9y, \qquad (3)$$

$$3x = 7 + 4y. \qquad (4)$$

Divide (3) by 2 and (4) by 3, $x = \dfrac{11+9y}{2}$, (5)

$$x = \frac{7+4y}{3}. \qquad (6)$$

Equate the values of x, $\dfrac{11+9y}{2} = \dfrac{7+4y}{3}$. (7)

Reduce (7) $33 + 27y = 14 + 8y$.

$$\therefore y = -1.$$

Substitute the value of y in (5), $\therefore x = 1$.

In this solution x is eliminated by *comparison*.

Each equation must be simplified, if necessary, before the elimination is performed.

(4) Solve: $\left.\begin{aligned}(x-1)(y+2)&=(x-3)(y-1)+8\\ \frac{2x-1}{5}-\frac{3(y-2)}{4}&=1\end{aligned}\right\}$ $\begin{matrix}(1)\\(2)\end{matrix}$

Simplify (1), $xy+2x-y-2=xy-x-3y+3+8.$

Transpose and combine, $3x+2y=13.$ (3)

Simplify (2), $8x-4-15y+30=20.$

Transpose and combine, $8x-15y=-6.$ (4)

Multiply (3) by 8, $24x+16y=104.$ (5)

Multiply (4) by 3, $24x-45y=-18.$ (6)

Subtract (6) from (5), $61y=122.$

$\therefore y=2.$

Substitute the value of y in (3), $3x+4=13.$

$\therefore x=3.$

Fractional simultaneous equations, with denominators which are simple expressions containing the unknown numbers, may be solved as follows:

(5) Solve: $\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7\\ \frac{7}{6x}-\frac{1}{10y}&=3\end{aligned}\right\}$ $\begin{matrix}(1)\\(2)\end{matrix}$

Multiply (2) by 4, $\frac{14}{3x}-\frac{2}{5y}=12.$ (3)

Add (1) and (3), $\frac{19}{3x}=19.$

Divide both sides by 19, $\frac{1}{3x}=1.$

$\therefore x=\frac{1}{3}.$

Substitute the value of x in (1),

$$5+\frac{2}{5y}=7.$$

Transpose, $\frac{2}{5y}=2.$

Divide both sides by 2, $\frac{1}{5y}=1.$

$\therefore y=\frac{1}{5}.$

94. Literal Simultaneous Equations. The method of solving literal simultaneous equations is as follows:

Ex. Solve:
$$\left.\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

To find the value of y:

Multiply (1) by c, $\quad acx + bcy = cm \qquad (3)$

Multiply (2) by a, $\quad acx + ady = an \qquad (4)$

Subtract (4) from (3), $\quad (bc - ad)y = cm - an$

Divide by coefficient of y, $\quad y = \dfrac{cm - an}{bc - ad}.$

To find the value of x:

Multiply (1) by d, $\quad adx + bdy = dm \qquad (5)$

Multiply (2) by b, $\quad bcx + bdy = bn \qquad (6)$

Subtract (6) from (5), $\quad (ad - bc)x = dm - bn$

Divide by coefficient of x, $\quad x = \dfrac{dm - bn}{ad - bc}.$

95. If three simultaneous equations are given, involving three unknown numbers, one of the unknown numbers must be eliminated between *two pairs* of the equations; then a second between the resulting equations.

Likewise, if four or more equations are given, involving four or more unknown numbers, one of the unknown numbers must be eliminated between three or more pairs of the equations; then a second between pairs of the resulting equations; and so on.

Solve:
$$\left.\begin{aligned} 2x - 3y + 4z &= 4 \\ 3x + 5y - 7z &= 12 \\ 5x - y - 8z &= 5 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\\(3)\end{aligned}$$

Eliminate z between two pairs of these equations.

Multiply (1) by 2,	$4x - 6y + 8z = 8$	(4)
(3) is	$5x - y - 8z = 5$	
Add,	$9x - 7y = 13$	(5)
Multiply (1) by 7,	$14x - 21y + 28z = 28$	
Multiply (2) by 4,	$12x + 20y - 28z = 48$	
Add,	$26x - y = 76$	(6)
Multiply (6) by 7,	$182x - 7y = 532$	(7)
(5) is	$9x - 7y = 13$	
Subtract (5) from (7),	$173x = 519$	
	$\therefore x = 3.$	
Substitute the value of x in (6),	$78 - y = 76.$	
	$\therefore y = 2.$	

Substitute the values of x and y in (1),

$$6 - 6 + 4z = 4.$$
$$\therefore z = 1.$$

Exercise 10.

Solve the following sets of equations:

1. $\left.\begin{array}{r} 6x + 5y = 46 \\ 10x + 3y = 66 \end{array}\right\}.$

2. $\left.\begin{array}{r} 2x + 7y = 52 \\ 3x - 5y = 16 \end{array}\right\}.$

3. $\left.\begin{array}{r} 4x + 9y = 79 \\ 7x - 17y = 40 \end{array}\right\}.$

4. $\left.\begin{array}{r} 2x - 7y = 8 \\ 4y - 9x = 19 \end{array}\right\}.$

5. $\left.\begin{array}{l} x = 16 - 4y \\ y = 34 - 4x \end{array}\right\}.$

6. $\left.\begin{array}{l} 5x = 2y + 78 \\ 3y = x + 104 \end{array}\right\}.$

7. $\left.\begin{array}{l} \dfrac{2x}{3} + \dfrac{y}{2} = 10 \\ \dfrac{y}{4} = \dfrac{5x - 7}{19} \end{array}\right\}.$

8. $\left.\begin{array}{l} 4 + y = \dfrac{3x}{4} \\ x - 8 = \dfrac{4y}{5} \end{array}\right\}.$

9. $\left.\begin{aligned}\frac{x+y}{3}+x&=15\\ \frac{x-y}{5}+y&=\ \ 6\end{aligned}\right\}.$

10. $\left.\begin{aligned}\frac{x-1}{8}+\frac{y-2}{5}&=2\\ \frac{2x}{7}+\frac{2y-5}{21}&=3\end{aligned}\right\}.$

11. $\left.\begin{aligned}\frac{3}{x}+\frac{8}{y}&=3\\ \frac{15}{x}-\frac{4}{y}&=4\end{aligned}\right\}.$

12. $\left.\begin{aligned}\frac{4}{5x}+\frac{5}{6y}&=\frac{86}{15}\\ \frac{5}{4x}-\frac{4}{5y}&=\frac{11}{20}\end{aligned}\right\}.$

13. $\left.\begin{aligned}\frac{x-2}{5}-\frac{10-x}{3}&=\frac{y-10}{4}\\ \frac{2y+4}{3}&=\frac{4x+y+13}{8}\end{aligned}\right\}.$

14. $\left.\begin{aligned}x-\frac{2y-x}{23-x}&=20+\frac{2x-59}{2}\\ y-\frac{y-3}{x-18}&=30-\frac{73-3y}{3}\end{aligned}\right\}.$

15. $\left.\begin{aligned}2x-3y&=5a-b\\ 3x-2y&=5a+b\end{aligned}\right\}.$

16. $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=1-\frac{x}{c}\\ \frac{x}{b}+\frac{y}{a}&=1+\frac{y}{c}\end{aligned}\right\}.$

17. $\left.\begin{aligned}\frac{x+y}{x-y}&=\frac{a}{b-c}\\ \frac{x+c}{y+b}&=\frac{a+b}{a+c}\end{aligned}\right\}.$

18. $\left.\begin{aligned}\frac{x-a}{y-a}&=\frac{a-b}{a+b}\\ \frac{x}{y}&=\frac{a^3-b^3}{a^3+b^3}\end{aligned}\right\}.$

19. $\left.\begin{aligned}8x+4y-3z&=6\\ x+3y-\ \ z&=7\\ 4x-5y+4z&=8\end{aligned}\right\}.$

20. $\left.\begin{aligned}3x-\frac{y}{4}+z\ \ &=7\tfrac{1}{2}\\ 2x-\frac{y-3z}{3}&=5\tfrac{1}{3}\\ 2x-\frac{y}{2}+4z&=11\end{aligned}\right\}.$

21. $\left.\begin{aligned}\frac{3y-2x}{3z-7}&=\frac{1}{2}\\ \frac{5z-x}{2y-3z}&=1\\ \frac{y-2z}{3y-2x}&=1\end{aligned}\right\}.$

22. $\left.\begin{aligned} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}&=3 \\ \frac{a}{x}+\frac{b}{y}-\frac{c}{z}&=1 \\ \frac{2a}{x}&=\frac{b}{y}+\frac{c}{z} \end{aligned}\right\}.$

23. $\left.\begin{aligned} \frac{xy}{x+y}&=a \\ \frac{xz}{x+z}&=b \\ \frac{yz}{y+z}&=c \end{aligned}\right\}.$

96. Problems. It is often necessary in the solution of problems to employ two or more letters to represent the numbers to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown numbers employed.

If there are *more* equations than unknown numbers, some of them are superfluous or inconsistent; if there are *less* equations than unknown numbers, the problem is indeterminate.

Ex. If A gives B \$10, B will have three times as much money as A. If B gives A \$10, A will have twice as much money as B. How much has each?

Let x = number of dollars A has,
and y = number of dollars B has.

Then $y + 10$ = number of dollars B has, and $x - 10$ = number of dollars A has after A gives \$10 to B.

$\therefore\ y + 10 = 3(x - 10)$, and $x + 10 = 2(y - 10)$.

From the solution of these equations, $x = 22$ and $y = 26$.

Therefore A has \$22 and B has \$26.

Exercise 11.

1. Three times the greater of two numbers exceeds twice the less by 27; and the sum of twice the greater and five times the less is 94. Find the numbers.

2. A fraction is such that if 3 be added to each of its terms, the resulting fraction is equal to $\frac{4}{5}$; and if 3 be subtracted from each of its terms, the result is equal to $\frac{1}{2}$. Find the fraction.

3. Two women buy velvet and silk. One buys $3\frac{1}{2}$ yards of velvet and $12\frac{3}{4}$ yards of silk; the other took $4\frac{1}{2}$ yards of velvet and 5 yards of silk. They each pay $63.80. How much per yard did each cost?

4. Each of two persons owes $1200. The first said to the second, "If you give me $\frac{3}{4}$ of what you have, I shall have enough to pay my debt." The second replied, "If you give me $\frac{8}{9}$ of what your purse contains, I can pay my debt." How much does each have?

5. Two passengers have together 400 pounds of baggage. One pays $1.20, the other $1.80, for excess above the weight allowed. If all the baggage had belonged to one person he would have had to pay $4.50. How much baggage is allowed free?

6. A number is formed by two digits. The sum of the digits is 6 times their difference. The number itself exceeds 6 times the sum of its digits by 3. Find the number.

7. A number is formed by two digits of which the sum is 8. If the digits be interchanged, 4 times the new number exceeds the original number by 2 more than 5 times the sum of the digits. Find the original number.

8. Three brothers, A, B, C, have together bought a house for $32,000. A could pay the whole sum if B would give him $\frac{5}{8}$ of what he has; B could pay it if C would give him $\frac{8}{9}$ of what he has; and C could pay the whole sum if he had $\frac{1}{2}$ of what A has together with $\frac{3}{16}$ of what B has. How much does each have?

9. A and B entered into partnership with a joint capital of $3400. A put in his money for 12 months; B put in his money for 16 months. In closing the business, A's share of the profits was greater than B's by $\frac{5}{59}$ of the total profit. Find the sum put in by each.

10. A capitalist makes two investments; the first at 3 per cent, the second at $3\frac{1}{2}$ per cent. His total income from the two investments is \$427. If \$1400 were taken from the second investment and added to the first, the incomes from the two investments would be equal. Find the amount of each investment.

11. A cask contains 12 gallons of wine and 18 gallons of water; a second cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be taken from each cask, so that, when mixed, there may be 14 gallons consisting half of water and half of wine?

12. A and B ran a race to a post and back. A returning meets B 30 yards from the post and beats him by 1 minute. If on arriving at the starting place A had immediately returned to meet B, he would have run $\frac{1}{6}$ the distance to the post before meeting him. Find the distance run, and the time A and B each makes.

13. A and B together can do a piece of work in 15 days. After working together for 6 days, A leaves off and B finishes the work in 30 days more. In how many days can each do the work?

14. A and B together can do a piece of work in 12 days. After working together 9 days, however, they call in C to aid them, and the three finish the work in 2 days. C finds that he can do as much work in 5 days as A does in 6 days. In how many days can each do the work?

15. A pedestrian has a certain distance to walk. After having passed over 20 miles, he increases his speed by 1 mile per hour. If he had walked the entire journey with this speed, he would have accomplished his walk in 40 minutes less time; but, by keeping his first pace, he would have arrived 20 minutes later than he did. What distance had he to walk?

CHAPTER VII.

INVOLUTION AND EVOLUTION.

97. The operation of raising an expression to any required *power* is called **Involution.**

Every case of involution is merely an example of *multiplication,* in which the factors are *equal.*

98. Index Law. If m is a positive integer, by definition

$$a^m = a \times a \times a \cdots\cdots \text{ to } m \text{ factors.} \qquad \S\ 13$$

Consequently, if m and n are both positive integers,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots\cdots \text{ to } m \text{ factors}\\ &= (a \times a \cdots\cdots \text{ to } n \text{ factors})(a \times a \cdots\cdots \text{ to } n \text{ factors})\\ &\qquad \ldots\ldots \text{ taken } m \text{ times}\\ &= a \times a \times a \cdots\cdots \text{ to } mn \text{ factors}\\ &= a^{mn}.\end{aligned}$$

The above is the **index law** for involution.

Also,

$$(a^m)^n = a^{mn} = (a^n)^m;$$

$$\begin{aligned}(ab)^n &= ab \times ab \cdots\cdots \text{ to } n \text{ factors}\\ &= (a \times a \cdots\cdots \text{ to } n \text{ factors})(b \times b \cdots\cdots \text{ to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

99. If the exponent of the required power is a composite number, the exponent may be resolved into prime factors, the power denoted by one of these factors found, and the result raised to a power denoted by another factor of the exponent. Thus, the fourth power may be obtained by taking the second power of the second power; the sixth by taking the second power of the third power; and so on.

100. From the *Law of Signs* in multiplication it is evident that all *even* powers of a number are *positive;* all *odd* powers of a number have the *same sign* as the number itself.

Hence, no *even* power of *any* number can be *negative;* and the even powers of two compound expressions which have the same terms with opposite signs are identical.

Thus, $\quad (b-a)^2 = \{-(a-b)\}^2 = (a-b)^2.$

101. Binomials. By actual multiplication we obtain,

$$(a+b)^2 = a^2 + 2ab + b^2;$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In these results it will be observed that:

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. b appears in the second term with 1 for an exponent, and its exponent increases by 1 in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying the coefficient of that term by the exponent of a, and dividing the product by a number greater by one than the exponent of b.

If b is negative, the terms in which the *odd* powers of b occur are negative. Thus,

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

By the above rules any power of a binomial of the form $a \pm b$ may be written at once.

102. The same method may be employed when the terms of a binomial have *coefficients* or *exponents*.

(1) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

(2) $(5x^2-2y^3)^3$,
$$= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3,$$
$$= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9.$$

In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its term in parentheses, so as to give the expression the form of a binomial.

(3) $(x^3-2x^2+3x+4)^2$,
$$= \{(x^3-2x^2)+(3x+4)\}^2,$$
$$= (x^3-2x^2)^2 + 2(x^3-2x^2)(3x+4) + (3x+4)^2,$$
$$= x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16,$$
$$= x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16.$$

Exercise 12.

In the following expressions perform the indicated operations:

1. $(2a^3)^4$.
2. $(3a^2x^3)^3$.
3. $\left(\dfrac{2a^4b^2}{3c^3x^4}\right)^5$.
4. $(-4b^2c)^3$.
5. $(-a^2b^3c)^4$.
6. $\dfrac{(3a^4b^2)^4}{(9a^6b^3)^3}$.
7. $(-5a^3b^2x^4)^3$.
8. $(6a^2b^3c^4)^4$.
9. $\dfrac{(-3a^3x^2)^5}{(6a^5bx^2)^3}$.
10. $\dfrac{(3a^2x^3)^2(4b^2x)^4}{(6b^3x^5)^2(a^2b)^2}$.
11. $\dfrac{(4x^4y)^3}{(9x^2y^3)^4} \div \dfrac{(x^2y^5)^2}{(3y)^5}$.
12. $(x+3)^3$.
13. $(1-2x)^4$.
14. $(3-x^2)^5$.
15. $(1-4x)^6$.
16. $\left(1-\dfrac{2x}{3}\right)^5$.
17. $\left(1+\dfrac{3x^2}{4}\right)^4$.
18. $\left(\dfrac{3}{2x}-\dfrac{2x}{3}\right)^5$.
19. $(1+3x)^7$.
20. $\left(\dfrac{2}{x}-\dfrac{3x}{4}\right)^6$.

21. $\left(2a^2b^m - \frac{a^mb^2}{2}\right)^5$.

22. $\left(x^{n-2} - \frac{y^{n-4}}{4x^n}\right)^4$.

23. $(1 + 3x - x^2)^4$.

24. $\left(1 + \frac{x}{2} - \frac{x^2}{4}\right)^3$.

103. The nth **root** of a number is one of the n equal factors of that number.

The operation of finding any required root of an expression is called **Evolution.**

Every case of evolution is merely an example of *factoring*, in which the required factors are all *equal*. Thus, the square, cube, fourth..... roots of an expression are found by taking one of the *two*, *three*, *four*..... *equal factors* of the expression.

The symbol which denotes that a square root is to be extracted is $\sqrt{\ }$; and for other roots the same symbol is used, but with a figure written above to indicate the root; thus, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., signifies the *third* root, *fourth* root, etc.

104. Index Law. If m and n are positive integers we have (§ 98),

$$(a^m)^n = a^{mn}.$$

Consequently $$\sqrt[n]{a^{mn}} = a^m.$$

Hence, the root of a simple expression is found by *dividing the exponent of each factor by the index of the root, and taking the product of the resulting factors.*

Thus, the *cube root* a^6 is a^2; the *fourth root* of $81a^{12}$ is $3a^3$; and so on.

The above is the **index law** for evolution.

Also, since $$(ab)^n = a^nb^n,$$

therefore, $$\sqrt[n]{a^nb^n} = ab = \sqrt[n]{a^n}\,\sqrt[n]{b^n}.$$

105. It is evident from § 100 that

I. Any *even* root of a *positive* number will have the double sign, $\pm$.

II. There can be no *even* root of a *negative* number.

Thus, $\sqrt{-x^2}$ is neither $+x$ nor $-x$, for $(+x)^2 = +x^2$, and $(-x)^2 = +x^2$.

The indicated even root of a negative number is called an **impossible**, or **imaginary**, number.

III. Any *odd* root of a number will have the same sign as the number.

106. Square Roots of Compound Expressions. Since the square of $a+b$ is $a^2+2ab+b^2$, the square root of

$$a^2+2ab+b^2 \text{ is } a+b.$$

It is required to devise a method of extracting the square root $a+b$ when $a^2+2ab+b^2$ is given.

$$\begin{array}{r|l}
 & a^2+2ab+b^2 \,\underline{|\, a+b} \\
 & a^2 \\
\hline
2a+b & 2ab+b^2 \\
 & \underline{2ab+b^2}
\end{array}$$

Ex. The first term, a, of the root is obviously the square root of the first term, a^2, in the expression.

If the a^2 be subtracted from the given expression, the remainder is $2ab+b^2$. Therefore the second term, b, of the root is obtained by dividing the first term of this remainder by $2a$, that is, by *double the part of the root already found.* Also, since $2ab+b^2 = (2a+b)b$, the divisor *is completed by adding to the trial-divisor the new term of the root.*

The same method will apply to longer expressions, if care be taken to obtain the *trial-divisor* at each stage of the process, by *doubling the part of the root already found,* and to obtain the *complete divisor by annexing the new term of the root to the trial-divisor.*

Find the square root of

$$1 + 10x^2 + 25x^4 + 16x^6 - 24x^5 - 20x^3 - 4x.$$

$$
\begin{array}{r|l}
 & 16x^6 - 24x^5 + 25x^4 - 20x^3 + 10x^2 - 4x + 1 \,\big|\, \underline{4x^3 - 3x^2 + 2x - 1} \\
 & \underline{16x^6} \\
8x^3 - 3x^2 & -24x^5 + 25x^4 \\
 & \underline{-24x^5 + 9x^4} \\
8x^3 - 6x^2 + 2x & 16x^4 - 20x^3 + 10x^2 \\
 & \underline{16x^4 - 12x^3 + 4x^2} \\
8x^3 - 6x^2 + 4x - 1 & -8x^3 + 6x^2 - 4x + 1 \\
 & \underline{-8x^3 + 6x^2 - 4x + 1}
\end{array}
$$

The expression is arranged according to descending powers of x.

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its *last term doubled*.

107. Arithmetical Square Roots. In the general method of extracting the square root of a number expressed by figures, the first step is to mark off the figures in *periods*.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any number expressed by *one* or *two* figures is a number of *one* figure; the square root of any number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, a dot be placed over the *units' figure* of a square number, and over every *alternate* figure, the number of dots will be equal to the number of figures in its square root.

Find the square root of 3249.

$$
\begin{array}{rl}
 & 3\dot{2}4\dot{9}\,(57 \\
 & \underline{25} \\
107) & 749 \\
 & \underline{749}
\end{array}
$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is a^2, and the complete divisor $2a + b$ is $2 \times 50 + 7 = 107$.

The same method will apply to numbers of more than two periods by considering a in the typical form to represent at each step *the part of the root already found.*

108. If the square root of a number has decimal places, the number itself will have *twice* as many.

Thus, if 0.21 be the square root of some number, this number will be $(0.21)^2 = 0.21 \times 0.21 = 0.0441$; and if 0.111 be the root, the number will be $(0.111)^2 = 0.111 \times 0.111 = 0.012321$.

Therefore, the number of *decimal* places in every square decimal will be *even*, and the number of decimal places in the root will be *half* as many as in the given number itself.

Hence, if the given square number contains a decimal, and a dot be placed over the *units' figure*, and then over every *alternate* figure on *both* sides of it, the number of dots to the left of the decimal point will show the number of *integral* places in the root, and the number of dots to the right will show the number of *decimal* places.

Ex. Find the square roots of 41.2164 and 965.9664.

$$\begin{array}{rl} & 4\dot{1}.2\dot{1}6\dot{4}(6.42 \\ & 36 \\ \hline 124) & 521 \\ & 496 \\ \hline 1282) & 2564 \\ & 2564 \\ \hline \end{array} \qquad \qquad \begin{array}{rl} & \dot{9}6\dot{5}.9\dot{6}6\dot{4}(31.08 \\ & 9 \\ \hline 61) & 65 \\ & 61 \\ \hline 6208) & 49664 \\ & 49664 \\ \hline \end{array}$$

It is seen from the dotting that the root of the first example will have one integral and two decimal places, and that the root of the second example will have two integral and two decimal places.

109. If a number contains an *odd* number of decimal places, or gives a *remainder* when as many figures in the root have been obtained as the given number has periods, then its exact square root cannot be found. We may, however, approximate to the root as near as we please by annexing ciphers and continuing the operation.

When a number contains an odd number of decimal places, we separate the decimal part into periods by placing a dot over the *second* decimal figure from the decimal point; another over the *fourth* figure from the decimal point; and so on.

Ex. Find the square roots of 3 and 357.357.

$$\begin{array}{rl}
& \dot{3}.(1.732\ldots\ldots \\
& 1 \\
27) & 200 \\
& 189 \\
343) & 1100 \\
& 1029 \\
3462) & 7100 \\
& 6924
\end{array}
\qquad\qquad
\begin{array}{rl}
& \dot{3}5\dot{7}.3\dot{5}7\dot{0}(18.903\ldots\ldots \\
& 1 \\
28) & 257 \\
& 224 \\
369) & 3335 \\
& 3321 \\
37803) & 147000 \\
& 113409
\end{array}$$

Exercise 13.

Simplify:

1. $\sqrt{16a^8b^4}$.
2. $\dfrac{\sqrt[3]{27a^9b^6}}{\sqrt{81a^4}}$.
3. $\sqrt[4]{81a^8b^{12}}$.
4. $\dfrac{\sqrt[4]{625x^8}}{\sqrt[3]{64x^6}}$.
5. $\sqrt[5]{1024a^{10}b^5}$.
6. $\dfrac{\sqrt[3]{216a^6x^3}}{\sqrt[5]{32a^{15}x^{10}}}$.

Extract the square root of:

7. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
8. $9 - 24x - 68x^2 + 112x^3 + 196x^4$.
9. $4 - 12x + 5x^2 + 26x^3 - 29x^4 - 10x^5 + 25x^6$.
10. $36x^2 - 120a^2x - 12a^4x + 100a^4 + 20a^6 + a^8$.
11. $4 + 9y^2 - 20x + 25x^2 + 30xy - 12y$.
12. $4x^4 + 9y^6 - 12x^2y^3 + 16x^2 + 16 - 24y^3$.
13. $\dfrac{x^4}{4} + \dfrac{y^6}{9} + \dfrac{x^2}{4} - \dfrac{x^2y^3}{3} + \dfrac{1}{16} - \dfrac{y^3}{6}$.

Extract to four places of decimals the square root of:

14. 326.
15. 1020.
16. 3.666.
17. 1.31213.

110. Cube Roots of Compound Expressions. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the cube root of

$$a^3+3a^2b+3ab^2+b^3 \text{ is } a+b.$$

It is required to devise a method for extracting the cube root $a+b$ when $a^3+3a^2b+3ab^2+b^3$ is given:

(1) Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

$$\begin{array}{l|l|l}
 & a^3+3a^2b+3ab^2+b^3 & \underline{a+b} \\
3a^2 & a^3 & \\
\quad +3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
\end{array}$$

The first term a of the root is obviously the cube root of the first term a^3 of the given expression.

If a^3 be subtracted, the remainder is $3a^2b+3ab^2+b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of a.*

Also, since $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)b$, the *complete divisor* is obtained by adding $3ab+b^2$ to the *trial-divisor* $3a^2$.

The same method may be applied to longer expressions by considering a in the typical form $3a^2+3ab+b^2$ to represent at each stage of the process *the part of the root already found.*

111. Arithmetical Cube Root. In extracting the cube root of a number expressed by figures, the first step is to mark it off into periods.

Since $1=1^3$, $1000=10^3$, $1,000,000=100^3$, and so on, it follows that the cube root of any number between 1 and 1000, that is, of any number which has *one, two,* or *three* figures, is a number of *one* figure; and that the cube root of any number between 1000 and 1,000,000, that is, of any number which has *four, five,* or *six* figures, is a number of *two* figures; and so on.

Hence, if a dot be placed over every *third* figure of a cube number, beginning with the *units' figure,* the number of dots will be equal to the number of figures in its cube root.

If the cube root of a number contains any decimal figures, the number itself will contain *three times* as many.

Hence, if the given cube number have decimal places, and a dot be placed over the *units' figure* and over every *third* figure on *both* sides of it, the number of dots to the *left* of the decimal point will show the number of *integral* figures in the root; and the number of dots to the *right* will show the number of *decimal* figures in the root.

If the given number is not a perfect cube, zeros may be annexed, and a value of the root may be found as near to the *true* value as we please.

112. It is to be observed that if a denotes the first term of the root, and b the second term of the root, the *first complete divisor* is,

$$3a^2 + 3ab + b^2,$$

and the *second trial-divisor* is $3(a+b)^2$, that is,

$$3a^2 + 6ab + 3b^2,$$

which may be obtained from the preceding complete divisor by adding to it *its second term and twice its third term.*

Ex. Extract the cube root of 5 to five places of decimals.

	5.000(1.70997
	1
$3 \times 10^2 = 300$	4000
$3(10 \times 7) = 210$	
$7^2 = 49$	
559	3913
259	87000000
$3 \times 1700^2 = 8670000$	
$3(1700 \times 9) = 45900$	
$9^2 = 81$	
8715981	78443829
45981	85561710
$3 \times 1709^2 = 8762043$	78858387
	67033230
	61334301

After the first two figures of the root are found, the next trial divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor to be employed being three times the square of the part of the root already found.

Since the fourth power is the square of the square, and the sixth power the square of the cube, the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root*. In like manner, the eighth, ninth, twelfth..... roots may be found.

Exercise 14.

Extract the cube root of:

1. $27 - 108x + 144x^2 - 64x^3.$

2. $x^6 - 3x^5 + 5x^3 - 3x - 1.$

3. $a^3 - a^2b + \dfrac{ab^2}{3} - \dfrac{b^3}{27}.$

4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6.$

5. $27 + 296x^3 - 125x^6 - 108x + 9x^2 - 15x^4 - 300x^5.$

6. $12x^2 - \dfrac{125}{x^3} - 54x - 59 + \dfrac{135}{x} + 8x^3 + \dfrac{75}{x^2}.$

7. $8x^3 - 36ax^2 + \dfrac{a^6}{x^3} + \dfrac{33a^4}{x} + 66a^2x - \dfrac{9a^5}{x^2} - 63a^3.$

Extract to three places of decimals the cube roots of:

8. 517. **9.** 1637. **10.** 3.25. **11.** 20.911.

CHAPTER VIII.

EXPONENTS.

113. Positive Integral Exponents. If n is a positive integer, we have, by definition,

$$a^n = a \times a \times a \cdots\cdots \text{ to } n \text{ factors.}$$

From this definition we have obtained the following laws, which hold true for positive integral exponents:

$$\text{I.}\quad a^m \times a^n = a^{m+n}. \qquad \S\ 32$$

$$\text{II.}\quad \frac{a^m}{a^n} = a^{m-n} \text{ if } m > n, \qquad \S\ 41$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} \text{ if } n > m.$$

$$\text{III.}\quad (a^m)^n = a^{mn}. \qquad \S\ 98$$

$$\text{IV.}\quad \sqrt[n]{a^{mn}} = a^m. \qquad \S\ 104$$

$$\text{V.}\quad (ab)^n = a^n b^n. \qquad \S\ 98$$

From law I., and the commutative and associative principles, laws II.–V. may readily be obtained.

114. In the case of fractional and negative exponents, we proceed as follows:

We *assume* laws I. and V. to hold for such exponents, and then proceed to investigate what meaning of fractional and negative exponents is consistent with these laws.

It being *assumed* that laws I. and V. hold true, it is easily *proved* that laws II., III., and IV. must hold true.

115. Positive Fractional Exponents. If n is a positive integer, $\frac{1}{n}$ is a positive fraction.

We have, by III.,

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n}\times n} = a^1 = a;$$

but $$(\sqrt[n]{a})^n = a. \quad \therefore a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Again, if m and n are both positive integers, by III.,

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n}\times n} = a^m;$$

but $$(\sqrt[n]{a^m})^n = a^m. \quad \therefore a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence, in a fractional exponent, the numerator indicates a power, and the denominator a root.

116. Negative Integral Exponents. Dividing a^3 successively by a in the ordinary manner, we have the series

$$a^3, \quad a^2, \quad a, \quad 1, \quad \frac{1}{a}, \quad \frac{1}{a^2}, \quad \frac{1}{a^3}. \qquad (1)$$

Dividing again by a by subtracting 1 from the exponent of the dividend, we have, since II. holds true, the series

$$a^3, \quad a^2, \quad a^1, \quad a^0, \quad a^{-1}, \quad a^{-2}, \quad a^{-3}. \qquad (2)$$

Comparing (1) and (2), we see that

$$a^0 = 1, \quad a^{-1} = \frac{1}{a}, \quad a^{-2} = \frac{1}{a^2}, \quad a^{-3} = \frac{1}{a^3}.$$

117. Negative Fractional Exponents. If n is a positive integer, $-\frac{1}{n}$ is a negative fraction, and we have

$$(a^{-\frac{1}{n}})^n = a^{-\frac{1}{n}\times n} = a^{-1} = \frac{1}{a};$$

but $$\left(\frac{1}{\sqrt[n]{a}}\right)^n = \frac{1}{(\sqrt[n]{a})^n} = \frac{1}{a}. \quad \therefore a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}} = \frac{1}{a^{\frac{1}{n}}}.$$

Again, if m and n are both positive integers, by III.,

$$(a^{-\frac{m}{n}})^n = a^{-\frac{m}{n}\times n} = a^{-m} = \frac{1}{a^m};$$

but $$\left(\frac{1}{\sqrt[n]{a^m}}\right)^n = \frac{1}{(\sqrt[n]{a^m})^n} = \frac{1}{a^m}. \quad \therefore a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}} = \frac{1}{a^{\frac{m}{n}}}.$$

Hence, whether the exponent be integral or fractional, we have always $a^{-m} = \dfrac{1}{a^m}$.

118. From the application of these laws, we obtain:

$$(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}; \quad (ab)^{\frac{p}{q}} = a^{\frac{p}{q}}b^{\frac{p}{q}};$$

$$\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}} = \sqrt[n]{a}\,\sqrt[n]{b};$$

and so on.

119. Compound expressions are multiplied and divided as follows:

(1) Multiply $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

$$\begin{array}{lllll}
x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} & & & & \\
x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} & & & & \\
\hline
x & + x^{\frac{3}{4}}y^{\frac{1}{4}} & + x^{\frac{1}{2}}y^{\frac{1}{2}} & & \\
 & - x^{\frac{3}{4}}y^{\frac{1}{4}} & - x^{\frac{1}{2}}y^{\frac{1}{2}} & - x^{\frac{1}{4}}y^{\frac{3}{4}} & \\
 & & + x^{\frac{1}{2}}y^{\frac{1}{2}} & + x^{\frac{1}{4}}y^{\frac{3}{4}} & + y \\
\hline
x & & + x^{\frac{1}{2}}y^{\frac{1}{2}} & & + y.
\end{array}$$

(2) Divide $\sqrt[3]{x^2} + \sqrt[3]{x} - 12$ by $\sqrt[3]{x} - 3$.

$$\begin{array}{l|l}
x^{\frac{2}{3}} + x^{\frac{1}{3}} - 12 & x^{\frac{1}{3}} - 3 \\
\cline{2-2}
\underline{x^{\frac{2}{3}} - 3x^{\frac{1}{3}}} & x^{\frac{1}{3}} + 4. \\
\quad + 4x^{\frac{1}{3}} - 12 & \\
\quad \underline{+ 4x^{\frac{1}{3}} - 12} &
\end{array}$$

Exercise 15.

1. Express with radical signs and positive exponents:

$$a^{\frac{4}{5}};\ b^{\frac{3}{7}};\ c^{-\frac{1}{3}};\ x^{-\frac{3}{5}};\ (y^{\frac{4}{7}})^{-3}.$$

2. Express with fractional exponents:

$$\sqrt[3]{a^4};\ \sqrt[5]{b^{12}};\ \frac{1}{c\sqrt[5]{c^4}};\ \sqrt{\frac{1}{x^{\frac{3}{5}}}};\ \frac{1}{\sqrt[5]{y^3}}.$$

3. Express with positive exponents:

$$(a^{-3})^5;\ \sqrt[4]{b^{-3}};\ (\sqrt{c})^{-\frac{3}{5}};\ \left(\frac{1}{\sqrt[4]{x^{-5}}}\right)^{-2}.$$

4. Express with negative exponents and without denominators:

$$\frac{a^2}{(4x)^3};\ \frac{a^{\frac{1}{2}}}{\sqrt{5x^3}};\ \frac{4x^{-3}}{3y^{-2}};\ \frac{2\sqrt{a^{-3}}}{3\sqrt[3]{x^5}}.$$

Simplify:

5. $a^{\frac{1}{2}} \times a^{-\frac{2}{3}} \times a^{-\frac{4}{5}};\ b^{\frac{2}{3}} \times b^{\frac{4}{5}}\sqrt[4]{b^{-3}};\ (\sqrt{c})^3\sqrt[3]{c^{-4}}.$

6. $a^{\frac{3}{4}} \times a^{\frac{1}{2}} \times \sqrt[3]{a^4};\ b\sqrt[3]{c} \div (cx)^{\frac{1}{4}};\ (a^{\frac{1}{4}}\sqrt[3]{ax})^{\frac{1}{6}}.$

7. $(3a)^{\frac{1}{4}}\sqrt{(16x)^3};\ \left(\frac{16a^{-4}}{81x^3}\right)^{-\frac{3}{4}};\ \left(\frac{9a^4}{16x^{-3}}\right)^{-\frac{3}{2}};\ \left(\frac{27a^3}{\sqrt{9x^4}}\right)^{-\frac{4}{3}}.$

Multiply:

8. $x^{\frac{4}{3}} - x^{\frac{2}{3}} + 1$ by $x^{\frac{2}{3}} + 1$.

9. $x^{2p} + x^p y^p + y^{2p}$ by $x^{2p} - x^p y^p + y^{2p}$.

10. $8a^{\frac{3}{7}} + 4a^{\frac{2}{7}}b^{-\frac{1}{7}} + 5a^{\frac{1}{7}}b^{-\frac{2}{7}} + 9b^{-\frac{3}{7}}$ by $2a^{\frac{4}{7}} - b^{-\frac{4}{7}}$.

Divide:

11. $x^{5n} + y^{5n}$ by $x^n + y^n$.

12. $x - y^{-4}$ by $x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-1} + x^{\frac{1}{4}}y^{-2} - y^{-3}$.

13. $a^{\frac{3}{7}} + b + c^{-\frac{1}{2}} - 3a^{\frac{1}{7}}b^{\frac{1}{3}}c^{-\frac{1}{6}}$ by $a^{\frac{1}{7}} + b^{\frac{1}{3}} + c^{-\frac{1}{6}}$.

RADICAL EXPRESSIONS.

120. An indicated root that cannot be exactly obtained is called a **surd**, or **irrational number.** An indicated root that can be exactly obtained is said to have the *form* of a surd.

The required root shows the **order** of a surd; and surds are named quadratic, cubic, biquadratic, according as the second, third, or fourth roots are required.

The product of a rational factor and a surd factor is called a **mixed surd**; as, $3\sqrt{2}$, $b\sqrt{a}$.

When there is no rational factor outside of the radical sign, the surd is said to be **entire**; as, $\sqrt{2}$, $\sqrt{a}$.

121. Since $\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} = \sqrt[n]{abc}$, the product of two or more surds of the same order will be a radical expression of the same order, the number under the radical sign being the product of the numbers under the several radical signs.

122. In like manner, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$. That is,

A factor under the radical sign whose root can be taken, may, by having the root taken, be removed from under the radical sign.

Conversely, since $a\sqrt{b} = \sqrt{a^2b}$,

A factor outside the radical sign may be raised to the corresponding power and placed under it.

By $\sqrt[n]{a}$, where a is positive, is meant, in this chapter, the positive number which taken n times as a factor gives a for the product.

123. A surd is in its *simplest form* when the expression under the radical sign is integral and as small as possible.

Surds which, when reduced to the simplest form, have the same surd factor, are said to be **similar**.

Simplify $\sqrt[3]{108}$; $\sqrt[5]{7x^2y^7}$.

$$\sqrt[3]{108} = \sqrt[3]{27 \times 4} = 3\sqrt[3]{4}.$$
$$\sqrt[5]{7x^2y^7} = \sqrt[5]{7x^2y^2 \times y^5} = y\sqrt[5]{7x^2y^2}.$$

124. The product or quotient of two surds of *the same order* may be obtained by taking the product or quotient of the rational factors and the surd factors separately.

Thus, $\qquad 2\sqrt{5} \times 5\sqrt{7} = 10\sqrt{35}.$

Surds of *the same order* may be compared by expressing them as entire surds.

Compare $\frac{2}{3}\sqrt{7}$ and $\frac{3}{5}\sqrt{10}$.

$$\tfrac{2}{3}\sqrt{7} = \sqrt{\frac{28}{9}},$$
$$\tfrac{3}{5}\sqrt{10} = \sqrt{\frac{18}{5}}.$$
$$\sqrt{\frac{28}{9}} = \sqrt{\frac{140}{45}}, \text{ and } \sqrt{\frac{18}{5}} = \sqrt{\frac{162}{45}}.$$

As $\sqrt{\frac{162}{45}}$ is greater than $\sqrt{\frac{140}{45}}$, $\frac{3}{5}\sqrt{10}$ is greater than $\frac{2}{3}\sqrt{7}$.

125. The *order* of a surd may be changed by changing the *power* of the expression under the radical sign.

Thus, $\qquad \sqrt{5} = \sqrt[4]{25}$; $\sqrt[3]{c} = \sqrt[6]{c^2}$.

Conversely, $\quad \sqrt[4]{25} = \sqrt{5}$; $\sqrt[6]{c^2} = \sqrt[3]{c}$.

In this way, surds of *different orders* may be reduced to the *same order*, and may then be compared, multiplied, or divided.

(1) To compare $\sqrt{2}$ and $\sqrt[3]{3}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8};$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

$\therefore \sqrt[3]{3}$ is greater than $\sqrt{2}$.

(2) To multiply $\sqrt[3]{4a}$ by $\sqrt{6x}$.

$$\sqrt[3]{4a} = (4a)^{\frac{1}{3}} = (4a)^{\frac{2}{6}} = \sqrt[6]{(4a)^2} = \sqrt[6]{16a^2};$$
$$\sqrt{6x} = (6x)^{\frac{1}{2}} = (6x)^{\frac{3}{6}} = \sqrt[6]{(6x)^3} = \sqrt[6]{216x^3}.$$
$$\therefore\ \sqrt[3]{4a} \times \sqrt{6x} = \sqrt[6]{16a^2 \times 216x^3} = 2\sqrt[6]{54a^2x^3}.\ \textit{Ans.}$$

(3) To divide $\sqrt[3]{3a}$ by $\sqrt{6b}$.

$$\sqrt[3]{3a} = (3a)^{\frac{1}{3}} = (3a)^{\frac{2}{6}} = \sqrt[6]{(3a)^2} = \sqrt[6]{9a^2};$$
$$\sqrt{6b} = (6b)^{\frac{1}{2}} = (6b)^{\frac{3}{6}} = \sqrt[6]{(6b)^3} = \sqrt[6]{216b^3}.$$
$$\therefore\ \sqrt[3]{3a} \div \sqrt{6b} = \sqrt[6]{\frac{9a^2}{216b^3}} = \frac{1}{6b}\sqrt[6]{1944a^2b^3}.\ \textit{Ans.}$$

Exercise 16.

Express as entire surds:

1. $3\sqrt{5}$; $5\sqrt{32}$; $a^2b\sqrt{bc}$; $3y^2\sqrt[4]{x^3y}$; $a^3\sqrt[4]{a^3b^2}$.

2. $5abc\sqrt{abc^{-1}}$; $\frac{2}{7}\sqrt{\frac{91}{8}}$; $(x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}$.

Express as mixed surds:

3. $\sqrt[3]{160x^4y^7}$; $\sqrt[3]{54x^2y^3}$; $\sqrt[4]{64x^5y^6}$; $\sqrt[3]{1372a^{15}b^{16}}$.

Simplify:

4. $2\sqrt[4]{80a^5b^2c^6}$; $7\sqrt{396x}$; $\sqrt{1\frac{11}{16}}$; $\sqrt{3\frac{1}{8}}$; $\sqrt{\frac{3a^2bx}{4cy^3}}$.

5. $\left(\frac{x^3y^2}{z^2}\right)\left(\frac{z^5}{x^5y^5}\right)^{\frac{1}{2}}$; $\left(\frac{a^3b^2}{c^4}\right)\left(\frac{c^9b^3}{a}\right)^{\frac{1}{3}}$; $(2a^2b^4) \times (b^2x^3)^{\frac{1}{3}}$.

6. Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds.

7. Show that $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\frac{a^6}{b}}$ are similar surds.

8. Arrange in order of magnitude $9\sqrt{3}$, $6\sqrt{7}$, $5\sqrt{10}$.

9. Arrange in order of magnitude $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.

10. Multiply $3\sqrt{2}$ by $4\sqrt{6}$; $\frac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.

11. Divide $2\sqrt{5}$ by $3\sqrt{15}$; $\frac{3}{8}\sqrt{21}$ by $\frac{9}{16}\sqrt{\frac{7}{20}}$.

12. Simplify $\dfrac{2\sqrt{10}}{3\sqrt{27}} \times \dfrac{7\sqrt{48}}{5\sqrt{14}} \div \dfrac{4\sqrt{15}}{15\sqrt{21}}$.

Arrange in order of magnitude:

13. $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$.

14. $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

Simplify:

15. $\sqrt[4]{a^2xy^3} \times \sqrt[5]{a^3xy}$; $3\sqrt[3]{4ab^2} \div \sqrt{2a^3b}$.

16. $\sqrt{\left(\dfrac{16}{25}\right)^7} \times \sqrt{\left(\dfrac{25}{64}\right)^6}$; $(\sqrt[7]{a^3b})^3 \times \sqrt[7]{(a^3b^{12})^4}$.

126. In the addition or subtraction of surds, each surd must be reduced to its simplest form; then, if the resulting surds be similar,

Add the rational factors, and to their sum annex the common surd factor.

If the resulting surds be not similar,

Connect them with their proper signs.

127. Operations with surds will be more easily performed if the arithmetical numbers contained in the surds be expressed in their prime factors, and if fractional exponents be used instead of radical signs.

(1) Simplify $\sqrt{27} + \sqrt{48} + \sqrt{147}$.

$$\sqrt{27} = (3^3)^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}} = 3\sqrt{3};$$
$$\sqrt{48} = (2^4 \times 3)^{\frac{1}{2}} = 2^2 \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{1}{2}} = 4\sqrt{3};$$
$$\sqrt{147} = (7^2 \times 3)^{\frac{1}{2}} = 7 \times 3^{\frac{1}{2}} = 7\sqrt{3}.$$
$$\therefore \sqrt{27} + \sqrt{48} + \sqrt{147} = (3 + 4 + 7)\sqrt{3} = 14\sqrt{3}. \quad \textit{Ans.}$$

(2) Simplify $2\sqrt[3]{320} - 3\sqrt[3]{40}$.

$$2\sqrt[3]{320} = 2(2^6 \times 5)^{\frac{1}{3}} = 2 \times 2^2 \times 5^{\frac{1}{3}} = 8\sqrt[3]{5};$$
$$3\sqrt[3]{40} = 3(2^3 \times 5)^{\frac{1}{3}} = 3 \times 2 \times 5^{\frac{1}{3}} = 6\sqrt[3]{5}.$$
$$\therefore 2\sqrt[3]{320} - 3\sqrt[3]{40} = 8\sqrt[3]{5} - 6\sqrt[3]{5} = 2\sqrt[3]{5}. \quad \textit{Ans.}$$

128. If we wish to find the approximate value of $\frac{3}{\sqrt{2}}$, it will save labor if we multiply both numerator and denominator by a factor that will render the denominator *rational;* in this case by $\sqrt{2}$. Thus,

$$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

129. It is easy to rationalize the denominator of a fraction when that denominator is a *binomial* involving only quadratic surds. The factor required will consist of the terms of the given denominator, connected by a different sign. Thus, $\frac{7 - 3\sqrt{5}}{6 + 2\sqrt{5}}$ will have its denominator rationalized by multiplying both terms of the fraction by $6 - 2\sqrt{5}$. For,

$$\frac{7 - 3\sqrt{5}}{6 + 2\sqrt{5}} = \frac{(7 - 3\sqrt{5})(6 - 2\sqrt{5})}{(6 + 2\sqrt{5})(6 - 2\sqrt{5})}$$
$$= \frac{72 - 32\sqrt{5}}{16} = \frac{9}{2} - 2\sqrt{5}.$$

130. By two operations the denominator of a fraction may be rationalized when that denominator consists of *three* quadratic surds.

Thus, if the denominator be $\sqrt{6} + \sqrt{3} - \sqrt{2}$, both terms of the fraction may be multiplied by $\sqrt{6} - \sqrt{3} + \sqrt{2}$. The resulting denominator will be $6 - 5 + 2\sqrt{6} = 1 + 2\sqrt{6}$; and if both terms of the resulting fraction be multiplied by $1 - 2\sqrt{6}$, the denominator will become $1 - 24$ or -23.

Exercise 17.

Simplify:

1. $\sqrt{27}+2\sqrt{48}+3\sqrt{108}$; $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{432}$.

2. $2\sqrt{3}+3\sqrt{1\frac{1}{3}}-\sqrt{5\frac{1}{3}}$; $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}-\sqrt{\frac{3}{5}}$.

3. $\sqrt{\frac{a^4c}{b^3}}-\sqrt{\frac{a^2c^3}{bd^2}}-\sqrt{\frac{a^2cd^2}{bm^2}}$; $3\sqrt{\frac{2}{5}}+2\sqrt{\frac{1}{10}}-4\sqrt{\frac{1}{40}}$.

4. $2\sqrt[3]{40}+3\sqrt[3]{108}+\sqrt[3]{500}-\sqrt[3]{320}-2\sqrt[3]{1372}$.

5. $(\sqrt[9]{8})^4$; $(\sqrt[9]{27})^4$; $(\sqrt[8]{64})^3$; $(\sqrt[3]{4})^2$.

6. $(a\sqrt[3]{a})^{-3}$; $(x\sqrt[3]{x})^{-\frac{1}{2}}$; $(p^2\sqrt{p})^{\frac{1}{5}}$; $(a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{4}}$.

Find the square root of:

7. $x^{4m}+6x^{3m}y^n+11x^{2m}y^{2n}+6x^my^{3n}+y^{4n}$.

8. $1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2}$.

Simplify:

9. $\left(x^{\frac{2}{3}}\sqrt{\left(\frac{x^{\frac{1}{2}}}{\sqrt[3]{x}}\right)^5}\right)^{\frac{3}{26}}$.

10. $\left(\frac{\sqrt[5]{a^3}\sqrt{b}}{c\sqrt[4]{b^3}}\right)\times(a\sqrt[5]{ab^5})$.

11. $\left(\frac{\sqrt[4]{a^3}\sqrt[3]{b^2}}{5\sqrt[4]{c^5}}\right)\left(\frac{2b^{\frac{3}{2}}}{5a\sqrt[12]{b^2c^9}}\right)$.

12. $\left(\frac{10\sqrt[3]{a^2}}{\sqrt[4]{5b^{11}}}\right)\left(\frac{5a\sqrt[3]{a^2}}{4b\sqrt[5]{a^2}}\right)$.

13. $\left(\frac{x^{p+q}}{x^q}\right)^p\left(\frac{x^{q-p}}{x^q}\right)^{p-q}$.

14. $\frac{x^{2p(q+1)}-y^{2q(p-1)}}{x^{p(q+1)}+y^{q(p-1)}}$.

Find equivalent fractions with rational denominators for the following, and find their approximate values:

15. $\frac{3}{\sqrt{7}+\sqrt{5}}$; $\frac{7}{2\sqrt{5}-\sqrt{6}}$; $\frac{4-\sqrt{2}}{1+\sqrt{2}}$; $\frac{6}{5-2\sqrt{6}}$.

16. $\frac{2}{\sqrt{3}}$; $\frac{1}{\sqrt{5}-\sqrt{2}}$; $\frac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}$; $\frac{7-2\sqrt{3}+3\sqrt{2}}{3+3\sqrt{3}-2\sqrt{2}}$.

CHAPTER IX.

QUADRATIC EQUATIONS.

We now resume the subject of equations where we left it at the end of Chapter VI. Having considered equations of the first degree with one or more unknowns, we come next to the consideration of quadratic equations.

131. A quadratic equation which involves but one unknown number can contain only:

(1) Terms involving the square of the unknown number.

(2) Terms involving the first power of the unknown number.

(3) Terms which do not involve the unknown number.

Collecting similar terms, every quadratic equation can be made to assume the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are known numbers, and x the unknown number.

If a, b, c are given numbers, the equation is a **numerical quadratic**. If a, b, c are numbers represented wholly or in part by letters, the equation is a **literal quadratic**.

Thus, $x^2 - 6x + 5 = 0$ is a numerical quadratic,
and $ax^2 + 2bx + 3c - ab = 0$ is a literal quadratic.

132. In the equation $ax^2 + bx + c = 0$, a, b, and c are called the **coefficients** of the equation. The third term c is called the **constant** term.

If the first power of x is wanting, the equation is a **pure quadratic**; in this case, $b=0$.

If the first power of x is present, the equation is an **affected** or **complete quadratic.**

133. Solution of Pure Quadratic Equations:

(1) Solve the equation $5x^2-48=2x^2$.

We have	$5x^2-48=2x^2$.
Collect the terms,	$3x^2=48$.
Divide by 3,	$x^2=16$.
Extract the root,	$x=\pm 4$.

Observe that the roots are numerically equal, but one is positive and the other negative. There are but two roots, since there are but two square roots of any number.

It may seem as though we ought to write the sign $\pm$ before the x as well as before the 4. If we do this, we have

$$+x=+4,\quad -x=-4,\quad +x=-4,\quad -x=+4.$$

From the first and second, $x=4$; from the third and fourth, $x=-4$; these values of x are both given by $x=\pm 4$. Hence it is *unnecessary*, although *perfectly correct*, to write the $\pm$ sign on *both* sides of the reduced equation.

(2) Solve the equation $3x^2-15=0$.

We have	$3x^2=15$,
or	$x^2=5$.
Extract the root,	$x=\pm\sqrt{5}$.

The roots cannot be found exactly, since the square root of 5 cannot be found exactly; it can, however, be found as accurately as we please; for example, it lies between 2.23606 and 2.23607.

(3) Solve the equation $3x^2+15=0$.

We have	$3x^2=-15$,
or	$x^2=-5$.
Extract the root,	$x=\pm\sqrt{-5}$.

There is no square root of a negative number, since any number, positive or negative, multiplied by itself, gives a positive result.

The square root of -5 differs from the square root of $+5$ in that the latter can be found as accurately as we please, while the former cannot be found at all.

134. A root which can be found exactly is called an **exact** or **rational** root. Such roots are either whole numbers or fractions.

A root which is indicated but can be found only approximately is called a **surd** or **irrational** root. Such roots involve the roots of imperfect powers.

Exact and surd roots are together called **real** roots.

A root which is indicated but cannot be found, either exactly or approximately, is called an **imaginary** root. Such roots involve the even roots of negative numbers.

Exercise 18.

Solve:

1. $\dfrac{x^2-5}{3}+\dfrac{2x^2+1}{6}=\dfrac{1}{2}.$

2. $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8.$

3. $\dfrac{3}{4x^2}-\dfrac{1}{6x^2}=\dfrac{7}{3}.$

4. $5x^2-9=2x^2+24.$

5. $\dfrac{x^2}{5}-\dfrac{x^2-10}{15}=7-\dfrac{50+x^2}{25}.$

6. $\dfrac{3x^2-27}{x^2+3}+\dfrac{90+4x^2}{x^2+9}=7.$

7. $\dfrac{4x^2+5}{10}-\dfrac{2x^2-5}{15}=\dfrac{7x^2-25}{20}.$

8. $\dfrac{10x^2+17}{18}-\dfrac{12x^2+2}{11x^2-8}=\dfrac{5x^2-4}{9}.$

9. $x^2+bx+a=bx(1-bx).$

10. $ax^2+b=c.$

11. $x^2 - ax + b = ax(x-1)$.

12. $\dfrac{ab-x}{b-ax} = \dfrac{b-cx}{bc-x}$.

13. $\dfrac{3(x+a)}{4x-a} - \dfrac{2x+a}{2a+x} = 1$.

14. $\dfrac{3a}{x-5a} + \dfrac{x+4a}{x+3a} = \dfrac{7a^2+2ax-x^2}{(x-5a)(x+3a)}$.

15. $\dfrac{2(a+2b)}{a+2x} + \dfrac{a-2x}{a+b} = \dfrac{b^2}{(a+b)(a+2x)}$.

135. Solution of Affected Quadratic Equations:

Since $(x \pm b)^2$ is identical with $x^2 \pm 2bx + b^2$, it is evident that the expression $x^2 \pm 2bx$ lacks only the third term b^2 of being a perfect square.

This third term is the square of half the coefficient of x.

Every affected quadratic may be made to assume the form $x^2 \pm 2bx = c$, by dividing the equation through by the coefficient of x^2 (§ 131).

To **solve** such an equation:

The first step is to add to both members *the square of half the coefficient of x*. This is called completing the square.

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

(1) Solve the equation $x^2 - 8x = 20$.

We have	$x^2 - 8x = 20.$
Complete the square,	$x^2 - 8x + 16 = 36.$
Extract the root,	$x - 4 = \pm 6.$
Reduce,	$x = 4 + 6 = 10,$
	or $x = 4 - 6 = -2.$

The roots are 10 and -2.

We write the $\pm$ sign on only one side of the equation, for the reason given after the first example of § 133.

Verify by putting these numbers for x in the given equation:

$x = 10$,	$x = -2$,
$10^2 - 8(10) = 20$,	$(-2)^2 - 8(-2) = 20$,
$100 - 80 = 20$.	$4 + 16 = 20$.

(2) Solve the equation $\dfrac{x+1}{x-1} = \dfrac{4x-3}{x+9}$.

Free from fractions, $(x+1)(x+9) = (x-1)(4x-3)$.

Simplify, $3x^2 - 17x = 6$.

Divide by 3, $x^2 - \frac{17}{3}x = 2$.

Complete the square, $x^2 - \frac{17}{3}x + \left(\dfrac{17}{6}\right)^2 = \dfrac{361}{36}$.

Extract the root, $x - \dfrac{17}{6} = \pm\dfrac{19}{6}$.

Reduce, $x = \dfrac{17}{6} + \dfrac{19}{6} = \dfrac{36}{6} = 6$,

or $x = \dfrac{17}{6} - \dfrac{19}{6} = -\dfrac{2}{6} = -\dfrac{1}{3}$.

The roots are 6 and $-\dfrac{1}{3}$.

Verify by putting these numbers for x in the original equation:

$x = 6$.	$x = -\dfrac{1}{3}$.
$\dfrac{6+1}{6-1} = \dfrac{24-3}{6+9}$,	$\dfrac{-\frac{1}{3}+1}{-\frac{1}{3}-1} = \dfrac{-\frac{4}{3}-3}{-\frac{1}{3}+9}$,
$\dfrac{7}{5} = \dfrac{21}{15}$.	$\dfrac{\frac{2}{3}}{-\frac{4}{3}} = \dfrac{-\frac{13}{3}}{\frac{26}{3}}$,
	$-\dfrac{2}{4} = -\dfrac{13}{26}$.

136. When the coefficient of x^2 is not unity, we may proceed as in the preceding section, or we may complete the square by another method.

Since $(ax \pm b)^2$ is identical with $a^2x^2 \pm 2abx + b^2$, it is evident that the expression $a^2x^2 \pm 2abx$ lacks only the third term b^2 of being a perfect square.

This third term is the square of the quotient obtained by dividing the second term by twice the square root of the first term.

Every affected quadratic may be made to assume the form $a^2x^2 \pm 2abx = c$ (§ 131).

To **solve** such an equation:

The first step is to *complete the square;* to do this, *we divide the second term by twice the square root of the first term, square the quotient, and add the result to both members of the equation.*

The second step is to *extract the square root* of each member of the resulting equation.

The third step is to *reduce* the two resulting simple equations.

137. Numerical Quadratics are solved as follows:

(1) Solve the equation $16x^2 + 5x - 3 = 7x^2 - x + 45.$

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

Simplify, $\quad 9x^2 + 6x = 48.$

Complete the square, $\quad 9x^2 + 6x + 1 = 49.$

Extract the root, $\quad 3x + 1 = \pm 7.$

Reduce, $\quad 3x = -1 + 7 \text{ or } -1 - 7;$

$$3x = 6 \text{ or } -8.$$

$$\therefore x = 2 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 2 for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

$$16(2)^2 + 5(2) - 3 = 7(2)^2 - (2) + 45,$$

$$64 + 10 - 3 = 28 - 2 + 45,$$

$$71 = 71.$$

Verify by substituting $-2\frac{2}{3}$ for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

$$16\left(-\frac{8}{3}\right)^2 + 5\left(-\frac{8}{3}\right) - 3 = 7\left(-\frac{8}{3}\right)^2 - \left(-\frac{8}{3}\right) + 45,$$

$$\frac{1024}{9} - \frac{40}{3} - 3 = \frac{448}{9} + \frac{8}{3} + 45,$$

$$1024 - 120 - 27 = 448 + 24 + 405,$$

$$877 = 877.$$

(2) Solve the equation $3x^2 - 4x = 32$.

Since the exact root of 3, the coefficient of x^2, cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number*.

Multiply by 3, $\quad 9x^2 - 12x = 96.$

Complete the square, $\quad 9x^2 - 12x + 4 = 100.$

Extract the root, $\quad 3x - 2 = \pm 10.$

Reduce, $\quad 3x = 2 + 10$ or $2 - 10$;

$$3x = 12 \text{ or } -8.$$

$$\therefore x = 4 \text{ or } -2\tfrac{2}{3}.$$

Or, divide by 3, $\quad x^2 - \dfrac{4x}{3} = \dfrac{32}{3}.$

Complete the square, $\quad x^2 - \dfrac{4x}{3} + \dfrac{4}{9} = \dfrac{32}{3} + \dfrac{4}{9} = \dfrac{100}{9}.$

Extract the root, $\quad x - \dfrac{2}{3} = \pm\dfrac{10}{3}.$

$$\therefore x = \frac{2 \pm 10}{3},$$

$$= 4 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 4 for x in the original equation,

$$48 - 16 = 32,$$

$$32 = 32.$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation,

$$21\tfrac{1}{3} + (10\tfrac{2}{3}) = 32,$$

$$32 = 32.$$

(3) Solve the equation $-3x^2+5x=-2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is

$$3x^2-5x=2.$$

Multiply by 3, $$9x^2-15x=6.$$

Complete the square, $$9x^2-15x+\frac{25}{4}=\frac{49}{4}.$$

Extract the root, $$3x-\frac{5}{2}=\pm\frac{7}{2}.$$

Reduce, $$3x=\frac{5\pm 7}{2},$$

$$3x=6 \text{ or } -1.$$

$$\therefore\ x=2 \text{ or } -\frac{1}{3}.$$

Or, divide by 3, $$x^2-\frac{5x}{3}=\frac{2}{3}.$$

Complete the square, $$x^2-\frac{5x}{3}+\frac{25}{36}=\frac{49}{36}.$$

Extract the root, $$x-\frac{5}{6}=\pm\frac{7}{6}.$$

$$\therefore\ x=\frac{5\pm 7}{6},$$

$$=2 \text{ or } -\frac{1}{3}.$$

If the equation $3x^2-5x=2$ be multiplied by *four times the coefficient of* x^2, fractions will be avoided:

$$36x^2-60x=24.$$

Complete the square, $$36x^2-60x+25=49.$$

Extract the root, $$6x-5=\pm 7.$$

$$6x=5\pm 7,$$

$$6x=12 \text{ or } -2.$$

$$\therefore\ x=2 \text{ or } -\frac{1}{3}.$$

It will be observed that the number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2-5x=2$.

(4) Solve the equation $\frac{3}{5-x} - \frac{1}{2x-5} = 2.$

Simplify, $4x^2 - 23x = -30.$

Multiply by four times the coefficient of x^2, and add to each side the square of the coefficient of x,

$$64x^2 - (\ ) + (23)^2 = 529 - 480 = 49.$$

Extract the root, $8x - 23 = \pm 7.$

Reduce, $8x = 23 \pm 7;$

$$8x = 30 \text{ or } 16.$$

$$\therefore x = 3\tfrac{3}{4} \text{ or } 2.$$

If a trinomial is a perfect square, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(5) Solve the equation $72x^2 - 30x = -7.$

Since $72 = 2^3 \times 3^2$, if the equation be multiplied by 2, the coefficient of x^2 in the resulting equation, $144x^2 - 60x = -14$, will be a square number, and the term required to complete the square will be $\left(\frac{60}{24}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}.$ Hence, if the original equation be multiplied by 4×2, the coefficient of x^2 in the result will be a square number, and fractions will be avoided in the work.

Multiply the given equation by 8,

$$576x^2 - 240x = -56.$$

Complete the square, $576x^2 - (\ ) + 25 = -31.$

Extract the root, $24x - 5 = \pm\sqrt{-31}.$

Reduce, $24x = 5 \pm \sqrt{-31}.$

$$\therefore x = \tfrac{1}{24}(5 \pm \sqrt{-31}).$$

Note. In solving the following equations, care must be taken to select the method best adapted to the example under consideration.

Exercise 19.

Solve :

1. $x^2 - 2x = 15.$
2. $x^2 - 14x = -48.$
3. $x^2 - x = 12.$
4. $x^2 - 3x = 28.$

5. $x^2 - 13x + 42 = 0.$

6. $x^2 - 21x + 108 = 0.$

7. $2x^2 + x = 6.$

8. $4x^2 + 7x = 15.$

9. $3x^2 - 19x + 28 = 0.$

10. $4x^2 + 17x - 15 = 0.$

11. $6x^2 - x = 12.$

12. $5x^2 - \frac{28}{3}x + 4 = 0.$

13. $6x^2 - 7x + \frac{5}{3} = 0.$

14. $\frac{x^2+1}{17} + (x+1)(x+2) = 0.$

15. $(x-5)^2 + x^2 - 5 = 16(x+3).$

16. $\frac{x^2}{6} + \frac{3x-19}{3} = \frac{11+x}{3}.$

17. $\frac{2x^2-11}{2x+3} = \frac{x+1}{2}.$

18. $\frac{x+1}{x} + \frac{x}{6} = \frac{11}{2x}.$

19. $\frac{x^2-4}{3x} + \frac{2x}{5} = x + \frac{1-2x}{5}.$

20. $x + \frac{x+6}{x-6} = 2(x-2).$

21. $\frac{6}{2x-6} + \frac{1}{3-x} = \frac{8}{x}.$

22. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$

23. $\frac{x-6}{x-2} + \frac{x+5}{2x+1} = 1.$

24. $\frac{x-3}{x+4} + \frac{x-4}{2(x-1)} = \frac{1}{2}.$

25. $\frac{x+1}{x^2-4} + \frac{1-x}{x+2} = \frac{2}{5(x-2)}.$

26. $\frac{x-5}{x+3}+\frac{x-8}{x-3}=\frac{80}{x^2-9}+\frac{1}{2}.$

27. $\frac{1}{x-3}+\frac{7}{x+3}=\frac{14}{x^2-9}-\frac{x-4}{x+3}.$

28. $\frac{3x+5}{x+3}+\frac{x+3}{x-3}=\frac{x-1}{x^2-9}.$

29. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}.$

30. $\frac{2x-1}{x+1}+\frac{3x-1}{x+2}+\frac{7-x}{x-1}=4.$

31. $\frac{3x+2}{1-5x}+\frac{x-7}{1+5x}+\frac{6(x^2-x+1)}{25x^2-1}+5=0.$

32. $\frac{x+7}{9-4x^2}-\frac{1-x}{2x+3}=\frac{4}{2x-3}.$

33. $\frac{2x+1}{x+3}+\frac{2(x+1)}{x+2}=2\frac{1}{12}.$

138. Literal Quadratics are solved as follows:

(1) Solve the equation $ax^2+bx+c=0.$

Transpose c, $\quad ax^2+bx=-c.$

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2+(\ )+b^2=b^2-4ac.$$

Extract the root, $\quad 2ax+b=\pm\sqrt{b^2-4ac}.$

Reduce, $\quad 2ax=-b\pm\sqrt{b^2-4ac}.$

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

(2) Solve the equation $adx-acx^2=bcx-bd.$

Transpose bcx and change the signs,

$$acx^2+bcx-adx=bd.$$

Express the left member in *two terms*,

$$acx^2+(bc-ad)x=bd.$$

Multiply by $4ac$,

$$4a^2c^2x^2 + 4ac(bc - ad)x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + (\ ) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root, $\quad 2acx + (bc - ad) = \pm (bc + ad)$

Reduce,

$$2acx = -(bc - ad) \pm (bc + ad)$$
$$= 2ad \text{ or } -2bc.$$
$$\therefore x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

(3) Solve the equation $px^2 - px + qx^2 + qx = \dfrac{pq}{p+q}.$

Express the left member in *two terms*,

$$(p+q)x^2 - (p-q)x = \frac{pq}{p+q}.$$

Multiply by four times the coefficient of x^2,

$$4(p+q)^2x^2 - 4(p^2 - q^2)x = 4pq.$$

Complete the square,

$$4(p+q)^2x^2 - (\ ) + (p-q)^2 = p^2 + 2pq + q^2.$$

Extract the root, $\quad 2(p+q)x - (p-q) = \pm (p+q).$

Reduce,

$$2(p+q)x = (p-q) \pm (p+q),$$
$$= 2p \text{ or } -2q.$$
$$\therefore x = \frac{p}{p+q} \text{ or } -\frac{q}{p+q}.$$

Observe that the left member of the simplified equation must be expressed in two terms, simple or compound, the first term involving x^2, the second involving x.

Exercise 20.

Solve:

1. $x^2 - 2ax = 3a^2.$
2. $x^2 + 7a^2 = 8ax.$
3. $4x(x-a) + a^2 = b^2.$
4. $\dfrac{x^2}{2} - \dfrac{ax}{3} = 2a(x + 2a).$
5. $x^2 = ax + b.$
6. $\dfrac{(x+a)^2}{a^2} = \dfrac{(x-a)^2}{b^2}.$
7. $x^2 - \dfrac{x}{a} = \dfrac{3}{4a^2}.$
8. $x^2 - (a+b)x = -ab.$
9. $x^2 - \dfrac{m^2+n^2}{mn}x + 1 = 0.$

10. $\frac{2x(a-x)}{3a-2x}=\frac{a}{4}.$

11. $2x^2+\frac{ab}{2}=(a+b)x.$

12. $(x+m)^2+(x-m)^2=5mx.$

13. $ax^2+5a^2x+\frac{9a^3}{4}=0.$

14. $b(a-x)^2=(b-1)x^2.$

15. $\frac{x}{a-x}+\frac{a+b}{x}=\frac{a}{a-x}.$

16. $\frac{x^2-ab}{x-b}=\frac{x+a}{2}.$

17. $\frac{a+b}{x-2a}+\frac{2a+b}{a}=\frac{x}{a}.$

18. $\frac{ax}{b^2}+\frac{a+x}{x}=\frac{5a+x}{2b}.$

19. $\frac{ab}{ax-bx}=a+b-(a-b)x.$

20. $\frac{5ab-3b^2-ax}{2a-x}=\frac{2a+x}{3}.$

21. $x^2-ax=\frac{(3a+2x)b}{2}+\frac{3(a^2+b^2)}{4}.$

22. $\frac{3a}{x+a}+\frac{2a}{x+2a}=\frac{4a}{x}+\frac{a}{x+3a}.$

23. $\frac{a-b+x}{a+b+x}+\frac{a+b}{x+b}=2.$

24. $\frac{a+4b}{x+2b}-\frac{a-4b}{x-2b}=\frac{4b}{a}.$

25. $\frac{4(x+a)}{a+b}-\frac{3(a+b)}{x+a}=4.$

26. $\frac{(4a^2-9b^2)(x^2+1)}{4a^2+9b^2}=2x.$

27. $(3a^2+b^2)(x^2-x+1)=(a^2+3b^2)(x^2+x+1).$

28. $\frac{4a^2}{x+2}-\frac{b^2}{x-2}=\frac{4a^2-b^2}{x(4-x^2)}.$

29. $\frac{a+2b}{a-2b}=\frac{a^2}{(a-2b)x}-\frac{4b^2}{x^2}.$

30. $\dfrac{x+1}{c}-\dfrac{2}{cx}=\dfrac{x+2}{ax-bx}.$

31. $\dfrac{a-c}{x-a}-\dfrac{x-a}{a-c}=\dfrac{3b(x-c)}{(a-c)(x-a)}.$

32. $x(x+b^2-b)=ax(a+1)-(a+b)^2(a-b).$

33. $\dfrac{x}{2}+\dfrac{(4m^2-n^2)mn}{x}=\dfrac{4m^2+n^2}{2}.$

34. $\dfrac{x^2}{m+n}-\left(1+\dfrac{1}{mn}\right)x+\dfrac{1}{m}+\dfrac{1}{n}=0.$

35. $\dfrac{2ab}{3x+1}+\dfrac{(3x-1)b^2}{2x+1}=\dfrac{(2x+1)a^2}{3x+1}.$

36. $\dfrac{x+2a-4b}{2bx}-\dfrac{8b-7a}{ax-2bx}+\dfrac{x-4a}{2(ab-2b^2)}=0.$

37. $\dfrac{1}{a+2b}-\dfrac{x}{a^2-4b^2}+\dfrac{x-5b}{(a+2b)x}=\dfrac{x+19b-2a}{2bx-ax}.$

38. $\dfrac{a-2b}{x+2b}+\dfrac{2(x+4a+3b)}{x-5a+3b}=0.$

39. $\dfrac{x+3b}{8a^2-12ab}+\dfrac{3b}{4a^2-9b^2}=\dfrac{a+3b}{(2a+3b)(x-3b)}.$

40. $\dfrac{1}{2x^2+x-1}+\dfrac{1}{2x^2-3x+1}=\dfrac{a}{2bx-b}+\dfrac{2bx+b}{a-ax^2}.$

41. $\dfrac{1}{x}+\dfrac{4ax^2+3b(2-x)}{2ax^2+2a+3b}=2.$

42. $\dfrac{x-a}{2b(x+a)}+\dfrac{2(ab-ax+2b^2)}{a(x+a)^2}=\dfrac{1}{a}.$

43. $\dfrac{2ax+b}{ax+b}+\dfrac{2ax-b}{ax-b}=\dfrac{9b^2x^2+(4a^2-6b^2)x-(a^2+b^2)}{a^2x^2-b^2}.$

44. $\dfrac{x+a+b}{x-3a+b}+\dfrac{3(a+c)}{x+b+c}=2.$

139. Solutions by a Formula. Every affected quadratic may be reduced to the form $x^2+px+q=0$, in which p and q represent numbers, positive or negative, integral or fractional.

$$\text{Solve:} \quad x^2+px+q=0.$$

$$4x^2+(\ )+p^2=p^2-4q,$$

$$2x+p=\pm\sqrt{p^2-4q}.$$

$$\therefore\ x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

By this formula, the values of x in an equation of the form $x^2+px+q=0$, may be written at once. Thus, take the equation

$$3x^2-5x+2=0.$$

Divide by 3, $\quad x^2-\frac{5}{3}x+\frac{2}{3}=0.$

Here, $\quad p=-\frac{5}{3}$, and $q=\frac{2}{3}$.

$$\therefore\ x=\frac{5}{6}\pm\frac{1}{2}\sqrt{\frac{25}{9}-\frac{8}{3}},$$

$$=\frac{5}{6}\pm\frac{1}{6},$$

$$=1 \text{ or } \frac{2}{3}.$$

140. Solutions by Factoring. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved *by inspection* into factors.

In this case the roots are seen at once without completing the square.

(1) Solve $x^2+7x-60=0$.

Since $\quad x^2+7x-60=(x+12)(x-5)$,

the equation $\quad x^2+7x-60=0$

may be written $\quad (x+12)(x-5)=0.$

It will be observed that if *either* of the factors $x+12$ or $x-5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x+12=0$, or $x-5=0$.

$\therefore x=-12$, or $x=5$.

(2) Solve $x^2+7x=0$.

The equation $x^2+7x=0$

becomes $x(x+7)=0$,

and is satisfied if $x=0$, or if $x+7=0$.

$\therefore$ the roots are 0 and -7.

It will be observed that this method is easily applied to an equation *all* the terms of which contain x.

(3) Solve $2x^3-x^2-6x=0$.

The equation $2x^3-x^2-6x=0$

becomes $x(2x^2-x-6)=0$,

and is satisfied if $x=0$, or if $2x^2-x-6=0$.

By solving $2x^2-x-6=0$ the two roots 2 and $-\frac{3}{2}$ are found.

$\therefore$ the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(4) Solve $x^3+x^2-4x-4=0$.

The equation $x^3+x^2-4x-4=0$

becomes $x^2(x+1)-4(x+1)=0$,

$(x^2-4)(x+1)=0$.

$\therefore$ the roots of the equation are -1, 2, -2.

(5) Solve $x^3-2x^2-11x+12=0$.

By trial we find that 1 satisfies the equation, and is therefore a root (§ 84).

Divide by $x-1$; the given equation may be written

$$(x-1)(x^2-x-12)=0,$$

and is satisfied if $x-1=0$, or if $x^2-x-12=0$.

The roots are found to be 1, 4, -3.

(6) Solve the equation $x(x^2-9)=a(a^2-9)$.

If we put a for x, the equation is satisfied; therefore a is a root (§ 84).

Transpose all the terms to the left member, and divide by $x-a$.

The given equation may be written

$$(x-a)(x^2+ax+a^2-9)=0,$$

and is satisfied if $x-a=0$, or if $x^2+ax+a^2-9=0$.

The roots are found to be

$$a,\ \frac{-a+\sqrt{36-3a^2}}{2},\ \frac{-a-\sqrt{36-3a^2}}{2}.$$

Exercise 21.

Find all the roots of:

1. $(x-1)(x-2)(x^2-4x+8)=0.$
2. $(x^2-2x+2)(x^2-6x+7)=0.$
3. $x^3+27=0.$
4. $x^4-81=0.$
5. $x^3-27+4(x^2-9)=0.$
6. $x^4+9x^2-16(x^2+9)=0.$
7. $2x^3+3x^2-2x-3=0.$
8. $x^4-4x^3+8x^2-32x=0.$
9. $x^3-x-6=0.$
10. $x^3-6x^2+11x-6=0.$
11. $x^4-3x^3-8x^2+6x+4=0.$
12. $x^3+x^2-14x-24=0.$
13. $x^4-6x^3+9x^2+4x-12=0.$
14. $x(x-3)(x+1)=a(a-3)(a+1).$
15. $x(x-3)(x+1)=20.$
16. $(x-1)(x-2)(x-3)=24.$
17. $(x+2)(x-3)(x+4)=240.$
18. $(x+1)(x+5)(x-6)=96.$

141. Character of the Roots. Every quadratic equation can be made to assume the form $ax^2 + bx + c = 0$.

Solving this equation (§ 138, Ex. 1), we obtain for its two roots

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a},\quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are two roots, and but two roots, since there are two, and but two, square roots of the expression $b^2 - 4ac$.

As regards the character of the two roots, there are three cases to be distinguished:

I. $b^2 - 4ac$ *positive.* In this case the roots are *real* and *different.* That the roots are different appears by writing them as follows:

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},\quad -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a};$$

these expressions cannot possibly be equal since $b^2 - 4ac$ is not zero.

If $b^2 - 4ac$ is a perfect square, the roots are exact. If $b^2 - 4ac$ is not a perfect square, the roots are surds.

II. $b^2 - 4ac = 0$. In this case the two roots are *real* and *equal,* since they both become $-\frac{b}{2a}$.

III. $b^2 - 4ac$ *negative.* In this case the roots are *imaginary,* since they both involve the square root of a negative number.

If we write them in the form

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},\quad -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a},$$

we see that two imaginary roots of a quadratic cannot be equal, since $b^2 - 4ac$ is not zero. Also that they have the

same real part, $-\frac{b}{2a}$, and the imaginary parts the same with opposite signs; such expressions are called *conjugate imaginaries.*

The above cases may also be distinguished as follows:

I. $b^2 - 4ac > 0$, roots real and different;

II. $b^2 - 4ac = 0$, roots real and equal;

III. $b^2 - 4ac < 0$, roots imaginary.

142. By calculating the value of $b^2 - 4ac$ we can determine the character of the roots of a given equation without solving the equation.

Examples:

(1) $x^2 - 5x + 6 = 0.$

Here $a = 1, \quad b = -5, \quad c = 6.$
$b^2 - 4ac = 25 - 24 = 1.$

The roots are real and different, and exact.

(2) $3x^2 + 7x - 1 = 0.$

Here $a = 3, \quad b = 7, \quad c = -1.$
$b^2 - 4ac = 49 + 12 = 61.$

The roots are real and different, and are both surds.

(3) $4x^2 - 12x + 9 = 0.$

Here $a = 4, \quad b = -12, \quad c = 9.$
$b^2 - 4ac = 144 - 144 = 0.$

The roots are real and equal.

(4) $2x^2 - 3x + 4 = 0.$

Here $a = 2, \quad b = -3, \quad c = 4.$
$b^2 - 4ac = 9 - 32 = -23.$

The roots are both imaginary.

(5) Find the values of m for which the equation

$$2mx^2 + (5m+2)x + (4m+1) = 0$$

has its two roots equal.

Here $\qquad a = 2m, \quad b = 5m+2, \quad c = 4m+1.$

If the roots are to be equal, we must have $b^2 - 4ac = 0$, or

$$(5m+2)^2 - 8m(4m+1) = 0.$$

This gives $m = 2$ or $-\frac{2}{7}$.

For these values of m the equation becomes

$$4x^2 + 12x + 9 = 0, \text{ and } 4x^2 - 4x + 1 = 0,$$

each of which has its roots equal.

Exercise 22.

Determine, without solving, the character of the roots of each of the following equations:

1. $x^2 - 6x + 8 = 0.$
2. $x^2 - 4x + 2 = 0.$
3. $x^2 + 6x + 13 = 0.$
4. $4x^2 - 12x + 7 = 0.$
5. $5x^2 - 9x + 6 = 0.$
6. $16x^2 - 56x + 49 = 0.$
7. $3x^2 - 2x + 12 = 0.$
8. $2x^2 - 19x + 17 = 0.$
9. $9x^2 + 30x + 25 = 0.$
10. $17x^2 - 12x + \frac{36}{17} = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

11. $(3m+1)x^2 + (2m+2)x + m = 0.$
12. $(m-2)x^2 + (m-5)x + 2m - 5 = 0.$
13. $2mx^2 + x^2 - 6mx - 6x + 6m + 1 = 0.$
14. $mx^2 + 2x^2 + 2m = 3mx - 9x + 10.$

143. Problems involving Quadratics. Problems which involve quadratic equations apparently have two solutions, as a quadratic equation has two roots. When both roots are positive integers, they will give two solutions.

Fractional and negative roots will in some problems give solutions; in other problems they will not give solutions.

No difficulty will be found in selecting the result which belongs to the particular problem we are solving. Sometimes, by a change in the statement of the problem, we may form a new problem which corresponds to the result that was inapplicable to the original problem.

Imaginary roots indicate that the problem is impossible.

(1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,

and $x + 1 =$ the other.

Then, $x^2 + (x+1)^2 = 481,$

or $2x^2 + 2x + 1 = 481.$

The solution of which gives $x = 15$ or -16.

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4....

(2) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ number of eggs for a shilling.

Then, $\frac{1}{x} =$ cost of 1 egg in shillings,

and $\frac{12}{x} =$ cost of 1 dozen in shillings.

But if $x + 2 =$ number of eggs for a shilling,

$\frac{12}{x+2} =$ cost of 1 dozen in shillings.

$$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12} \text{ (1 penny being } \tfrac{1}{12} \text{ of a shilling).}$$

The solution of which gives $x = 16$ or -18.

And, if 16 eggs cost a shilling, 1 dozen will cost $\frac{12}{16}$ of a shilling, or 9 pence.

Therefore the price of the eggs is 9 pence per dozen.

If the problem be changed so as to read: What is the price of eggs per dozen when 2 *less* in a shilling's worth *raises* the price 1 penny per dozen? the algebraic statement will be

$$\frac{12}{x-2} - \frac{12}{x} = \frac{1}{12}.$$

The solution of which gives $x = 18$ or -16.

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result.

Exercise 23.

1. The product of two consecutive numbers exceeds their sum by 181. Find the numbers.

2. The square of the sum of two consecutive numbers exceeds the sum of their squares by 220. Find the numbers.

3. The difference of the cubes of two consecutive numbers is 817. Find the numbers.

4. The difference of two numbers is 5 times the less, and the square of the less is twice the greater. Find the numbers.

5. The numerator of a certain fraction exceeds the denominator by 1. If the numerator and denominator be interchanged, the sum of the resulting fraction and the original fraction is $2\frac{1}{30}$. What was the original fraction?

6. The denominator of a certain fraction exceeds twice the numerator by 3. If $3\frac{3}{14}$ be added to the fraction, the resulting fraction is the reciprocal of the original fraction. Find the original fraction.

7. A farmer bought a number of geese for \$24. Had he bought 2 more geese for the same money, he would have paid $\frac{2}{5}$ of a dollar less for each. How many geese did he buy, and what did he pay for each?

State the problem to which the negative solution applies.

8. A laborer worked a number of days, and received for his labor \$36. Had his wages been 20 cents more per day, he would have received the same amount for 2 days' less labor. What were his daily wages, and how many days did he work?

State the problem to which the negative solution applies.

9. For a journey of 336 miles, 4 days less would have sufficed had I travelled 2 miles more per day. How many days did the journey take?

State the problem to which the negative solution applies.

10. A farmer hires a number of acres for \$420. He lets all but 4 for \$420, and receives for each acre \$2.50 more than he pays for it. How many acres does he hire?

11. A broker sells a number of railway shares for \$3240. A few days later, the price having fallen \$9 per share, he buys, for the same sum, 5 more shares than he had sold. Find the number of shares transferred on each day, and the price paid.

12. A man bought a number of sheep for \$300. He kept 15, and sold the remainder for \$270, gaining half a dollar on each sheep sold. How many sheep did he buy, and what did he pay for each?

13. The length of a rectangular lot exceeds its breadth by 20 yards. If each dimension be increased by 20 yards, the area of the lot will be doubled. Find the dimensions of the lot.

14. Twice the breadth of a rectangular lot exceeds the length by 2 yards; the area of the lot is 1200 square yards. Find the length and breadth.

15. Three times the breadth of a rectangular field, of which the area is 2 acres, exceeds twice the length by 8 rods. At $5 per rod, what will it cost to fence the field?

16. Two pipes running together fill a cistern in $10\frac{2}{7}$ hours; the larger will fill the cistern in 6 hours less time than the smaller. How long will it take each, running alone, to fill the cistern?

17. Three workmen, A, B, and C, dig a ditch. A can dig it alone in 6 days more time, B in 30 days more time, than the time it takes the three to dig the ditch together; C can dig the ditch in 3 times the time the three dig it in. How many days does it take the three, working together, to dig the ditch?

18. A cistern holding 900 gallons can be filled by two pipes running together in as many hours as the larger pipe brings in gallons per minute; the smaller pipe brings in per minute one gallon less than the larger pipe. How long will it take each pipe by itself to fill the cistern?

19. A number is formed by two digits, the second being less by 3 than one-half the square of the first. If 9 be added to the number, the order of the digits will be reversed. Find the number.

20. A number is formed by two digits; 5 times the second digit exceeds the square of the first digit by 4. If 3 times the first digit be added to the number, the order of the digits will be reversed. Find the number.

21. A boat's crew row 3 miles down a river and back again in 1 hour and 15 minutes. Their rate in still water is 3 miles per hour faster than twice the rate of the current. Find the rates of the crew and the rate of the current.

22. A jeweller sold a watch for $22.75, and lost on the cost of the watch as many per cent as the watch cost dollars. What was the cost of the watch?

23. A farmer sold a horse for $138, and gained on the cost $\frac{1}{8}$ as many per cent as the horse cost dollars. Find the cost of the horse.

24. A broker bought a number of $100 shares, when they were a certain per cent below par, for $8500. He afterwards sold all but 20, when they were the same per cent above par, for $9200. How many shares did he buy, and what did he pay for each share?

25. A drover bought a number of sheep for $110; 4 having died, he sold the remainder for 7.33\frac{1}{3}$ a head, and made on his investment four times as many per cent as he paid dollars for each sheep bought. How many sheep did he buy, and how many dollars did he make?

26. A certain train leaves A for B, distant 216 miles; 3 hours later another train leaves A to travel over the same route; the second train travels 8 miles per hour faster than the first, and arrives at B 45 minutes behind the first. Find the time each train takes to travel over the route.

27. A coach, due at B twelve hours after it leaves A, after travelling from A as many hours as it travels miles per hour, breaks down; it then proceeds at a rate 1 mile per hour less than half its former rate, and arrives at B three hours late. Find the distance from A to B.

CHAPTER X.

SIMULTANEOUS QUADRATIC EQUATIONS.

Quadratic equations involving two unknown numbers require different methods for their solution, according to the form of the equations.

144. Case I. When from one of the equations the value of one of the unknown numbers can be found in terms of the other, and this value *substituted* in the other equation.

Ex. Solve:
$$\left.\begin{aligned} 3x^2 - 2xy &= 5 \\ x - y &= 2 \end{aligned}\right\} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Transpose x in (2), $\quad y = x - 2.$

Substitute in (1), $\quad 3x^2 - 2x(x-2) = 5.$

The solution of which gives $\quad x = 1$ or -5.

$\therefore y = -1$ or -7.

Special methods often give more elegant solutions than the general method by substitution.

I. *When equations have the form,* $x \pm y = a$, *and* $xy = b$; $x^2 \pm y^2 = a$, *and* $xy = b$; *or,* $x \pm y = a$, *and* $x^2 + y^2 = b$.

(1) Solve:
$$\left.\begin{aligned} x + y &= 40 \\ xy &= 300 \end{aligned}\right\} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Square (1), $\quad x^2 + 2xy + y^2 = 1600. \quad (3)$

Multiply (2) by 4, $\quad 4xy = 1200. \quad (4)$

Subtract (4) from (3), $\quad x^2 - 2xy + y^2 = 400. \quad (5)$

Extract root of each side, $\quad x - y = \pm 20. \quad (6)$

Add (6) and (1), $\quad 2x = 60$ or 20,

Subtract (6) from (1), $\quad 2y = 20$ or 60.

$$\therefore \left.\begin{aligned} x &= 30 \\ y &= 10 \end{aligned}\right\} \text{ or } \left.\begin{aligned} x &= 10 \\ y &= 30 \end{aligned}\right\}.$$

(2) Solve: $$\left.\begin{aligned} x - y &= 4 \\ x^2 + y^2 &= 40 \end{aligned}\right\}\qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (1), $x^2 - 2xy + y^2 = 16.$ (3)
Subtract (2) from (3), $-2xy = -24.$ (4)
Subtract (4) from (2), $x^2 + 2xy + y^2 = 64.$
Extract the root, $x + y = \pm 8.$ (5)
By combining (5) and (1), $\left.\begin{aligned} x &= 6 \\ y &= 2 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= -2 \\ y &= -6 \end{aligned}\right\}.$

(3) Solve: $$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{9}{20} \\ \frac{1}{x^2} + \frac{1}{y^2} &= \frac{41}{400} \end{aligned}\right\}\qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (1), $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{81}{400}.$ (3)

Subtract (2) from (3), $\frac{2}{xy} = \frac{40}{400}.$ (4)

Subtract (4) from (2), $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{400}.$

Extract the root, $\frac{1}{x} - \frac{1}{y} = \pm\frac{1}{20}.$ (5)

By combining (1) and (5), $\left.\begin{aligned} x &= 4 \\ y &= 5 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= 5 \\ y &= 4 \end{aligned}\right\}.$

II. *When one equation may be simplified by dividing it by the other.*

(4) Solve: $$\left.\begin{aligned} x^3 + y^3 &= 91 \\ x + y &= 7 \end{aligned}\right\}\qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13.$ (3)
Square (2), $x^2 + 2xy + y^2 = 49.$ (4)
Subtract (3) from (4), $3xy = 36.$
Divide by -3, $-xy = -12.$ (5)
Add (5) and (3), $x^2 - 2xy + y^2 = 1.$
Extract the root, $x - y = \pm 1.$ (6)
By combining (6) and (2), $\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}.$

145. Case II. When each of the two equations is *homogeneous* and of the *second degree*.

Solve:
$$\left.\begin{aligned} 2y^2 - 4xy + 3x^2 &= 17 \\ y^2 - x^2 &= 16 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Let $y = vx$, and substitute vx for y in both equations.

From (1), $2v^2x^2 - 4vx^2 + 3x^2 = 17$,

$$\therefore x^2 = \frac{17}{2v^2 - 4v + 3}.$$

From (2), $v^2x^2 - x^2 = 16$,

$$\therefore x^2 = \frac{16}{v^2 - 1}.$$

Equate the values of x^2,

$$\frac{17}{2v^2 - 4v + 3} = \frac{16}{v^2 - 1},$$
$$32v^2 - 64v + 48 = 17v^2 - 17,$$
$$15v^2 - 64v = -65.$$

The solution gives, $v = \frac{5}{3}$ or $\frac{13}{5}$.

$$v = \frac{5}{3},$$
$$y = vx = \frac{5x}{3}.$$

Substitute in (2),

$$\frac{25x^2}{9} - x^2 = 16,$$
$$x^2 = 9,$$
$$x = \pm 3,$$
$$y = \frac{5x}{3} = \pm 5.$$

$$v = \frac{13}{5},$$
$$y = vx = \frac{13x}{5}.$$

Substitute in (2),

$$\frac{169x^2}{25} - x^2 = 16,$$
$$x^2 = \frac{25}{9},$$
$$x = \pm\frac{5}{3},$$
$$y = \frac{13x}{5} = \pm\frac{13}{3}.$$

146. Case III. When the two equations are *symmetrical* with respect to x and y; that is, when x and y are similarly involved.

Thus, the expressions $2x^3 + 3x^2y^2 + 2y^3$, $2xy - 3x - 3y + 1$, $x^4 - 3x^2y - 3xy^2 + y^4$, are symmetrical expressions.

In this case the general rule is to combine the equations in such a manner as to remove the highest powers of x and y.

(1) Solve: $\left.\begin{aligned} x^3+y^3 &= 18xy \\ x+y &= 12 \end{aligned}\right\}$ (1) (2)

Divide (1) by (2), $x^2-xy+y^2=\dfrac{3xy}{2}$. (3)

To remove x^2 and y^2, square (2),

$$x^2+2xy+y^2=144. \qquad (4)$$

Subtract (4) from (3), $-3xy=\dfrac{3xy}{2}-144,$

which gives $xy=32.$

We now have, $\left.\begin{aligned} x+y &= 12 \\ xy &= 32 \end{aligned}\right\}.$

Solving as in Case I., we find, $\left.\begin{aligned} x &= 8 \\ y &= 4 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= 4 \\ y &= 8 \end{aligned}\right\}.$

(2) Solve: $\left.\begin{aligned} x^4+y^4 &= 337 \\ x+y &= 7 \end{aligned}\right\}$ (1) (2)

To remove x^4 and y^4, raise (2) to the fourth power,

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=2401,$$

Subtract (1), $x^4 \qquad\qquad +y^4 = 337,$

$$4x^3y+6x^2y^2+4xy^3=2064.$$

Divide by 2, $2x^3y+3x^2y^2+2xy^3=1032.$ (3)

Square (2) and multiply the result by $2xy$,

$$2x^3y+4x^2y^2+2xy^3=98xy. \qquad (4)$$

Subtract (4) from (3), $-x^2y^2=1032-98xy,$

or $x^2y^2-98xy=-1032.$

This is a quadratic equation, with xy for the unknown number.

Solving, we find, $xy=12$ or 86.

We now have to solve the two pairs of equations,

$$\left.\begin{aligned} x+y &= 7 \\ xy &= 12 \end{aligned}\right\}, \quad \left.\begin{aligned} x+y &= 7 \\ xy &= 86 \end{aligned}\right\}.$$

From the first, $\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\}$ or $\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\}.$

From the second, $\left.\begin{aligned} x &= \frac{7\pm\sqrt{-295}}{2} \\ y &= \frac{7\mp\sqrt{-295}}{2} \end{aligned}\right\}.$

The preceding cases are *general methods* for the solution of equations which belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may be found more easily than by the general method.

Exercise 24.

1. $\left.\begin{aligned} x+y&=8 \\ xy&=15 \end{aligned}\right\}.$

2. $\left.\begin{aligned} x+y&=6 \\ xy+27&=0 \end{aligned}\right\}.$

3. $\left.\begin{aligned} x-y&=5 \\ xy&=24 \end{aligned}\right\}.$

4. $\left.\begin{aligned} x-y&=16 \\ xy+60&=0 \end{aligned}\right\}.$

5. $\left.\begin{aligned} x+2y&=12 \\ xy&=18 \end{aligned}\right\}.$

6. $\left.\begin{aligned} 2x+3y&=1 \\ xy+15&=0 \end{aligned}\right\}.$

7. $\left.\begin{aligned} y&=9-3x \\ x^2&=10-xy \end{aligned}\right\}.$

8. $\left.\begin{aligned} x+2y&=12 \\ xy+y^2&=35 \end{aligned}\right\}.$

9. $\left.\begin{aligned} x-3y+9&=0 \\ xy-y^2+4&=0 \end{aligned}\right\}.$

10. $\left.\begin{aligned} x^2+y^2&=100 \\ x+y&=14 \end{aligned}\right\}.$

11. $\left.\begin{aligned} x^2+y^2&=17 \\ 4x+y&=15 \end{aligned}\right\}.$

12. $\left.\begin{aligned} 2x^2-y^2+8&=0 \\ 3x-y-2&=0 \end{aligned}\right\}.$

13. $\left.\begin{aligned} x^2+xy&=40 \\ 2x-3y&=1 \end{aligned}\right\}.$

14. $\left.\begin{aligned} x^2-y^2&=13 \\ 3x-2y&=9 \end{aligned}\right\}.$

15. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{5}{18} \\ xy&=54 \end{aligned}\right\}.$

16. $\left.\begin{aligned} \frac{1}{x}-\frac{1}{y}&=\frac{1}{36} \\ x-2y+15&=0 \end{aligned}\right\}.$

17. $\left.\begin{aligned} x^2+4y+11&=0 \\ 3x+2y+1&=0 \end{aligned}\right\}.$

18. $\left.\begin{aligned} x+3y+1&=0 \\ x+\frac{4y+1}{x+2y}&=2(y+1) \end{aligned}\right\}.$

19. $\left.\begin{aligned} x^2+y^2&=106 \\ xy&=45 \end{aligned}\right\}.$

20. $\left.\begin{aligned} x^2+y^2&=52 \\ xy+24&=0 \end{aligned}\right\}.$

21. $\left.\begin{aligned} x^2-xy&=3 \\ y^2+xy&=10 \end{aligned}\right\}.$

22. $\left.\begin{aligned} xy+y^2&=4 \\ 2x^2-y^2&=17 \end{aligned}\right\}.$

23. $\left.\begin{aligned} x^2+3xy&=27 \\ xy-y^2&=2 \end{aligned}\right\}.$

24. $\left.\begin{aligned} x^2+xy&=60 \\ y^2+xy&=40 \end{aligned}\right\}.$

25. $\left.\begin{aligned} x^2+2xy-y^2&=28 \\ 3x^2+2xy+2y^2&=72 \end{aligned}\right\}.$

26. $\left.\begin{aligned} x^2-4xy&=45 \\ y^2-xy&=6 \end{aligned}\right\}.$

27. $\left.\begin{aligned} x^2+3xy&=55 \\ 2y^2+xy&=18 \end{aligned}\right\}.$

28. $\left.\begin{aligned} x^2-xy+y^2&=37 \\ x^2+2xy+8&=0 \end{aligned}\right\}.$

29. $\left.\begin{aligned} x^2+xy+2y^2&=44 \\ 2x^2-xy+y^2&=16 \end{aligned}\right\}.$

30. $\left.\begin{aligned} 8x^2-3xy-y^2&=40 \\ 9x^2+xy+2y^2&=60 \end{aligned}\right\}.$

31. $\left.\begin{aligned} 3x^2+3xy+y^2&=52 \\ 5x^2+7xy+4y^2&=140 \end{aligned}\right\}.$

32. $\left.\begin{aligned} 4x^2+3xy+5y^2&=27 \\ 7x^2+5xy+9y^2&=47 \end{aligned}\right\}.$

33. $\left.\begin{aligned} 5x^2+3xy+2y^2&=188 \\ x^2-xy+y^2&=19 \end{aligned}\right\}.$

34. $\left.\begin{aligned} x^3+y^3&=65 \\ x+y&=5 \end{aligned}\right\}.$

35. $\left.\begin{aligned} x^3-y^3&=98 \\ x-y&=2 \end{aligned}\right\}.$

36. $\left.\begin{aligned} x^3+y^3&=279 \\ x+y&=3 \end{aligned}\right\}.$

37. $\left.\begin{aligned} x^3-y^3&=218 \\ x-y&=2 \end{aligned}\right\}.$

38. $\left.\begin{aligned} x^3+y^3&=152 \\ x^2-xy+y^2&=19 \end{aligned}\right\}.$

39. $\left.\begin{aligned} x^3-y^3&=1304 \\ x^2+xy+y^2&=163 \end{aligned}\right\}.$

40. $\left.\begin{aligned} x^3+y^3&=91 \\ xy(x+y)&=84 \end{aligned}\right\}.$

41. $\left.\begin{aligned} x^3-y^3&=98 \\ x-y&=\frac{30}{xy} \end{aligned}\right\}.$

42. $\left.\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x}&=\frac{27}{2} \\ \frac{1}{x}+\frac{1}{y}&=\frac{1}{2} \end{aligned}\right\}.$

43. $\left.\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x}&=\frac{91}{12} \\ \frac{1}{x}+\frac{1}{y}&=\frac{7}{12} \end{aligned}\right\}.$

44. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{1}{2} \\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{5}{36} \end{aligned}\right\}.$

45. $\left.\begin{aligned} x^3 - y^3 &= 7xy \\ x - y &= 2 \end{aligned}\right\}.$

46. $\left.\begin{aligned} x^3 + y^3 &= \frac{27xy}{2} \\ x + y &= 9 \end{aligned}\right\}.$

47. $\left.\begin{aligned} x^2 + y^2 &= \frac{5xy}{2} \\ x + y &= \frac{5xy}{6} \end{aligned}\right\}.$

48. $\left.\begin{aligned} x^2y^2 - 16xy + 60 &= 0 \\ x + y &= 7 \end{aligned}\right\}.$

49. $\left.\begin{aligned} x^2y^2 &= 4xy + 12 \\ xy &= x + y + 1 \end{aligned}\right\}.$

50. $\left.\begin{aligned} x^3 + y^3 &= \frac{35x^2y^2}{36} \\ x + y &= \frac{5xy}{6} \end{aligned}\right\}.$

51. $\left.\begin{aligned} x^2 + y^2 &= 67 - xy \\ x + y &= xy - 5 \end{aligned}\right\}.$

52. $\left.\begin{aligned} x^3 + y^3 &= 1 - 3xy \\ x^2 + y^2 &= xy + 37 \end{aligned}\right\}.$

53. $\left.\begin{aligned} x^4 + y^4 &= 706 \\ x + y &= 2 \end{aligned}\right\}.$

54. $\left.\begin{aligned} x^5 - y^5 &= 211 \\ x - y &= 1 \end{aligned}\right\}.$

55. $\left.\begin{aligned} x^5 + y^5 &= 3368 \\ x + y &= 8 \end{aligned}\right\}.$

56. $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{y^2}{x^2} &= 17\left(\frac{xy}{16}\right)^2 \\ \frac{1}{x} + \frac{1}{y} &= \frac{3}{4} \end{aligned}\right\}.$

57. $\left.\begin{aligned} x^2 + y^2 &= xy + 19 \\ x + y &= xy - 7 \end{aligned}\right\}.$

58. $\left.\begin{aligned} \frac{x+y}{x-y} + \frac{x-y}{x+y} &= \frac{10}{3} \\ x^2 + y^2 &= 45 \end{aligned}\right\}.$

59. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 133 \\ x^2 - xy + y^2 &= 19 \end{aligned}\right\}.$

60. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 931 \\ x^2 + xy + y^2 &= 49 \end{aligned}\right\}.$

61. $\left.\begin{aligned} x^2 + xy + y^2 &= 84 \\ x + \sqrt{xy} + y &= 6 \end{aligned}\right\}.$

62. $\left.\begin{aligned} x^2 + y^2 &= 819 - xy \\ x + y &= 21 + \sqrt{xy} \end{aligned}\right\}.$

63. $\left.\begin{aligned} x^4 + y^4 &= 97 \\ x^2 + y^2 &= 49 - x^2y^2 \end{aligned}\right\}.$

64. $\left.\begin{aligned} 2x^2 + 3xy + 12 &= 3y^2 \\ 3x + 5y + 1 &= 0 \end{aligned}\right\}.$

65. $\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{a}{x} + \frac{b}{y} &= 4 \end{aligned}\right\}.$

66. $\left.\begin{aligned} x + y &= a \\ 4xy &= a^2 - 4b^2 \end{aligned}\right\}.$

67. $\left.\begin{aligned} x^2 &= ax + by \\ y^2 &= bx + ay \end{aligned}\right\}.$

68. $\left.\begin{aligned} x^2 - xy &= a^2 + b^2 \\ xy - y^2 &= 2ab \end{aligned}\right\}.$

69. $\left.\begin{aligned} x^2 + y^2 + x + y &= 18 \\ xy &= 6 \end{aligned}\right\}.$

70. $\left.\begin{aligned} x^4+y^4 &= 10(x^2+y^2)+72 \\ 2(x^2+y^2) &= 5xy \end{aligned}\right\}.$

71. $\left.\begin{aligned} x^2+y^2 &= 2x^2y^2-15 \\ x+y &= xy+1 \end{aligned}\right\}.$

72. $\left.\begin{aligned} ay^2+bxy &= b \\ bx^2+axy &= a \end{aligned}\right\}.$

73. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{1}{x+y} \\ \frac{1}{x^2}+\frac{1}{y^2} &= \frac{1}{a^2} \end{aligned}\right\}.$

74. $\left.\begin{aligned} \frac{(x+y)^2}{a^2}+\frac{(x-y)^2}{b^2} &= 8 \\ x^2+y^2 &= 2(a^2+b^2) \end{aligned}\right\}.$

75. $\left.\begin{aligned} x^2+y^2-8ab &= 5(a^2+b^2) \\ xy-5ab &= 2(a^2+b^2) \end{aligned}\right\}.$

76. $\left.\begin{aligned} x^2+y^2 &= axy \\ x+y &= bxy \end{aligned}\right\}.$

77. $\left.\begin{aligned} 2(x^2+y^2) &= 5xy-9ab \\ 2(a+b)(x+y) &= 3(xy-ab) \end{aligned}\right\}.$

78. $\left.\begin{aligned} x^2+y^2+z^2 &= 49 \\ x+y+z &= 11 \\ 2x+3y-4z &= 6 \end{aligned}\right\}.$

79. $\left.\begin{aligned} xy+yz+xz &= 40 \\ 4x=3y &= 2z+4 \end{aligned}\right\}.$

80. $\left.\begin{aligned} x^2+y^2+z^2 &= 84 \\ x+y+z &= 14 \\ y^2 &= xz \end{aligned}\right\}.$

81. $\left.\begin{aligned} 2xy+x+y &= 22 \\ 2yz+y+z &= 58 \\ 2xz+x+z &= 32 \end{aligned}\right\}.$

82. $\left.\begin{aligned} x^2+xy+xz &= a^2 \\ y^2+yz+xy &= 2ab \\ z^2+xz+yz &= b^2 \end{aligned}\right\}.$

Exercise 25.

1. If the length and breadth of a rectangle were each increased 1 foot, the area would be 48 square feet; if the length and breadth were each diminished 1 foot, the area would be 24 square feet. Find the length and breadth of the rectangle.

2. A farmer laid out a rectangular lot containing 1200 square yards. He afterwards increased the width $1\frac{1}{2}$ yards and diminished the length 3 yards, thereby increasing the area by 60 square yards. Find the dimensions of the original lot.

3. The diagonal of a rectangle is 89 inches; if each side were 3 inches less, the diagonal would be 85 inches. Find the area of the rectangle.

4. The diagonal of a rectangle is 65 inches; if the rectangle were 3 inches shorter and 9 inches wider, the diagonal would still be 65 inches. Find the area of the rectangle.

5. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. Find the numbers.

6. The sum, the product, and the difference of the squares of two numbers are all equal. Find the numbers.

HINT. Represent the numbers by $x + y$ and $x - y$.

7. The sum of two numbers is 5, and the sum of their cubes is 335. Find the numbers.

8. The sum of two numbers is 11, and the cube of their sum exceeds the sum of their cubes by 792. Find the numbers.

9. A number is formed by two digits. The second digit is less by 8 than the square of the first digit; if 9 times

the first digit be added to the number, the order of the digits will be reversed. Find the number.

10. A number is formed by three digits, the third digit being the sum of the other two; the product of the first and third digits exceeds the square of the second by 5. If 396 be added to the number, the order of the digits will be reversed. Find the number.

11. The numerator and denominator of a certain fraction are each greater by 1 than those of a second fraction; the sum of the two fractions is $\frac{17}{12}$. If the numerators were interchanged, the sum of the fractions would be $\frac{3}{2}$. Find the fractions.

12. There are two fractions. The numerator of the first is the square of the denominator of the second, and the numerator of the second is the square of the denominator of the first; the sum of the fractions is $\frac{35}{6}$, and the sum of their denominators 5. Find the fractions.

13. The sum of two numbers which are formed by the same two digits is $\frac{55}{18}$ of their difference; the difference of the squares of the numbers is 3960. Find the numbers.

14. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; if the circumference of each were increased 2 feet, the fore wheel would turn only 88 times more. Find the circumference of each wheel.

15. Two travellers, A and B, set out from two distant towns, A to go from the first town to the second, and B from the second town to the first, and both travel at uniform rates. When they meet, A has travelled 30 miles farther than B. A finishes his journey 4 days, and B 9 days, after they meet. Find the distance between the towns, and the number of miles A and B each travel per day.

16. Two boys run in opposite directions around a rectangular field, of which the area is one acre; they start from one corner, and meet 13 yards from the opposite corner. One boy runs only $\frac{5}{6}$ as fast as the other. Find the length and breadth of the field.

17. A man walks from the base of a mountain to the summit, reaching the summit in $5\frac{1}{2}$ hours; during the last half of the distance he walks $\frac{1}{2}$ mile less per hour than during the first half. He descends in $3\frac{3}{4}$ hours, walking 1 mile per hour faster than during the first half of the ascent. Find the distance from the base to the summit and the rates of walking.

18. A besieged garrison had bread for 11 days. If there had been 400 more men, each man's daily share would have been 2 ounces less; if there had been 600 less men, each man's daily share could have been increased by 2 ounces, and the bread would then have lasted 12 days. How many pounds of bread did the garrison have, and what was each man's daily share?

19. Three students, A, B, and C, agree to work out a set of problems in preparation for an examination; each is to do all the problems. A solves 9 problems per day, and finishes the set 4 days before B; B solves 2 more problems per day than C, and finishes the set 6 days before C. Find the number of problems in the set.

20. A cistern can be filled by two pipes; one of these pipes can fill the cistern in 2 hours less time than the other; the cistern can be filled by both pipes running together in $1\frac{7}{8}$ hours. Find the time in which each pipe will fill the cistern.

21. A and B have a certain manuscript to copy between them. At A's rate of work he would copy the whole manuscript in 18 hours; B copies 9 pages per hour. A finishes his portion in as many hours as he copies pages per hour; B is occupied with his portion 2 hours longer than A is with his. Find the number of pages copied by each.

22. A and B have 4800 circulars to stamp, and intend to finish them in two days, 2400 each day. The first day A, working alone, stamps 800, and then A and B stamp the remaining 1600, A working altogether 3 hours. The second day A works 3 hours and B 1 hour, and they accomplish only $\frac{9}{10}$ of their task for that day. Find the number of circulars each stamps per minute, and the number of hours B works on the first day.

23. A, in running a race with B, to a post and back, meets him 10 yards from the post. To come in even with A, B must increase his pace from this point $41\frac{3}{7}$ yards per minute. If, without changing his pace, he turns back on meeting A, he will come in 4 seconds behind A. Find the distance to the post.

24. A boat's crew, rowing at half their usual speed, row 3 miles down stream and back again, accomplishing the distance in 2 hours and 40 minutes. At full speed they can go over the same course in 1 hour and 4 minutes. Find the rate of the crew and of the current.

25. A farmer sold a number of sheep for \$286. He received for each sheep \$2 more than he paid for it, and gained thereby on the cost of the sheep $\frac{1}{2}$ as many per cent as each sheep cost dollars. Find the number of sheep.

26. A person has \$1300, which he divides into two parts and loans at different rates of interest, in such a

manner that the two portions produce equal returns. If the first portion had been loaned at the second rate of interest it would have yielded annually \$36; if the second portion had been loaned at the first rate of interest it would have yielded annually \$49. Find the two rates of interest.

27. A person has \$5000, which he divides into two portions and loans at different rates of interest in such a manner that the return from the first portion is double the return from the second portion. If the first portion had been loaned at the second rate of interest it would have yielded annually \$245; if the second portion had been loaned at the first rate of interest it would have yielded annually \$90. Find the two amounts and the two rates of interest.

28. A number is formed by three digits; 10 times the middle digit exceeds the square of half the sum of the three digits by 21; if 99 be added to the number, the digits will be in reverse order; the number is 11 times the number formed by the first and third digit. Find the number.

29. A number is formed by three digits; the sum of the last two digits is the square of the first digit; the last digit is greater by 2 than the sum of the first and second; if 396 be added to the number, the digits will be in reverse order. Find the number.

30. A railroad train, after travelling 1 hour from A, meets with an accident which delays it 1 hour; it then proceeds at a rate 8 miles per hour less than its former rate, and arrives at B 5 hours late. If the accident had happened 50 miles further on, the train would have been only $3\frac{1}{3}$ hours late. Find the distance from A to B.

CHAPTER XI.

EQUATIONS SOLVED LIKE QUADRATICS.

147. Some equations not of the second degree may be solved by completing the square.

(1) Solve: $$8x^6 + 63x^3 = 8.$$

This equation is in the *quadratic form* if we regard x^3 as the unknown number.

We have, $$8x^6 + 63x^3 = 8.$$

Multiply by 32 and complete the square,

$$256x^6 + (\ ) + (63)^2 = 4225.$$

Extract the square root, $$16x^3 + 63 = \pm 65.$$

Hence, $$x^3 = \tfrac{1}{8} \text{ or } -8.$$

Extracting the cube root, two values of x are $\frac{1}{2}$ and -2. To find the remaining roots, it remains to solve completely the two equations

$$x^3 = \tfrac{1}{8}, \qquad x^3 = -8.$$

We have, $$8x^3 - 1 = 0,$$
or, $$(2x-1)(4x^2+2x+1) = 0.$$
$\therefore$ either $$2x - 1 = 0,$$
or, $$4x^2 + 2x + 1 = 0.$$

Solving these, we find for three values of x,

$$\tfrac{1}{2},\quad \frac{-1+\sqrt{-3}}{4},\quad \frac{-1-\sqrt{-3}}{4}.$$

We have, $$x^3 + 8 = 0,$$
or, $$(x+2)(x^2-2x+4) = 0.$$
$\therefore$ either $$x + 2 = 0,$$
or, $$x^2 - 2x + 4 = 0.$$

Solving these, we find for three values of x,

$$-2,\quad 1+\sqrt{-3},\quad 1-\sqrt{-3}.$$

These six values of x are the six roots of the given equation.

(2) Solve: $$\sqrt{x^3} - 3\sqrt[4]{x^3} = 40.$$

Using fractional exponents, we have $x^{\frac{3}{2}} - 3x^{\frac{3}{4}} = 40$.

This equation is in the quadratic form if we regard $x^{\frac{3}{4}}$ as the unknown number.

Complete the square, $4x^{\frac{3}{2}} - 12x^{\frac{3}{4}} + 9 = 169.$

Extract the root, $2x^{\frac{3}{4}} - 3 = \pm 13.$

$$\therefore 2x^{\frac{3}{4}} = 16 \text{ or } -10,$$

$$x^{\frac{3}{4}} = 8 \text{ or } -5,$$

$$x = 16 \text{ or } -5\sqrt[3]{5}.$$

There are other values of x which we shall not at present attempt to find.

Exercise 26.

Solve :

1. $x^6 + 7x^3 = 8.$
2. $x^4 - 5x^2 + 4 = 0.$
3. $x^6 + 4x^3 = 96.$
4. $37x^2 - 9 = 4x^4.$
5. $16x^8 = 17x^4 - 1.$
6. $32x^{10} = 33x^5 - 1.$
7. $x^6 + 14x^3 + 24 = 0.$
8. $19x^4 + 216x^7 = x.$
9. $x^8 - 22x^4 + 21 = 0.$
10. $x^{2m} + 3x^m = 4.$
11. $x^{4n} - \dfrac{5x^{2n}}{3} = \dfrac{25}{12}.$
12. $x^{6n} + 3x^{3n} = 40.$
13. $x^{2m} + 2ax^m = 8a^2.$
14. $x^{-4} - 4x^{-2} = 12.$
15. $x^{-6} + 5x^{-3} - 36 = 0.$
16. $x^{-8} - 3x^{-4} - 154 = 0.$
17. $9x^{-4} + 4x^{-2} = 5.$
18. $4x^{\frac{1}{2}} - 3x^{\frac{1}{4}} = 10.$
19. $2x^{\frac{1}{3}} - 3x^{\frac{1}{6}} = 9.$
20. $\sqrt{x^5} = \sqrt[4]{x^5} + 12.$
21. $x = 9\sqrt{x} + 22.$
22. $\sqrt[3]{x^2} - 4\sqrt[3]{x} = 32.$
23. $2\sqrt{x^3} - 3\sqrt[4]{x^3} = 35.$
24. $\dfrac{1}{\sqrt[3]{x}} + \dfrac{1}{\sqrt[6]{x}} = \dfrac{3}{4}.$
25. $x^{-\frac{1}{2}} + x^{-\frac{1}{4}} = \dfrac{4}{9}.$
26. $3x^{-\frac{1}{3}} + 4x^{-\frac{1}{6}} = 20.$
27. $2x^{-\frac{2}{3}} - x^{-\frac{1}{3}} = 45.$
28. $4\sqrt[3]{x^{-2}} + 3\sqrt[3]{x^{-1}} = 27.$
29. $\sqrt[3]{2x} + \sqrt[3]{4x^2} = 72.$
30. $\sqrt{2x} + 4x = 1.$

148. Radical Equations. If an equation involves radical expressions, we first clear of radicals as follows:

Solve $\sqrt{x+4}+\sqrt{2x+6}=\sqrt{7x+14}$.

Square both sides,

$$x+4+2\sqrt{(x+4)(2x+6)}+2x+6=7x+14.$$

Transpose and combine, $\quad 2\sqrt{(x+4)(2x+6)}=4x+4.$

Divide by 2 and square, $\quad (x+4)(2x+6)=(2x+2)^2.$

Multiply out and reduce, $\quad x^2-3x=10.$

Hence, $\quad x=5$ or -2.

Of these two values, only 5 will satisfy the equation as it stands, for -2 gives $\sqrt{2}+\sqrt{2}=0$.

The reason is that the square root sign simply indicates that *one* of the two square roots is to be taken, and does not indicate *which* square root. All we can expect is that *one* of the *two* possible square roots will cause the equation to be satisfied. Thus, in the preceding example, for $x=5$ we really have $\pm 3 \pm 4 = \pm 7$, which is true if the signs be taken either all positive or all negative.

Putting $x=-2$ we really have $\pm\sqrt{2}\pm\sqrt{2}=0$, which is true if one sign be taken positive and the other negative.

Since the square of either a positive or a negative number is positive, we see that the four equations obtained from

$$\pm\sqrt{x+4}\pm\sqrt{2x+6}=\sqrt{7x+14},$$

by taking the signs $++$, $+-$, $-+$, $--$, will all lead to the equation $x^2-3x=10$, and will consequently all give the same values of x; viz. 5 and -2.

149. Some radical equations may be solved as follows:

Solve $7x^2-5x+8\sqrt{7x^2-5x+1}=-8$.

Add 1 to both sides,

$$7x^2-5x+1+8\sqrt{7x^2-5x+1}=-7.$$

Put $\sqrt{7x^2-5x+1}=y$; the equation becomes

$$y^2+8y=-7.$$

Hence, $\quad y=-1$ or -7,

$$y^2=1 \text{ or } 49.$$

We now have $7x^2-5x+1=1$, or $7x^2-5x+1=49$.

Solving these, we find for the values of x,

$$0,\ \frac{5}{7} \;\Big|\; 3,\ \frac{16}{7}.$$

These values all satisfy the given equation when the radical is taken negative; they are in fact the four roots of the biquadratic obtained by clearing the given equation of radicals.

150. Various other equations may be solved by methods similar to that of the last section.

(1) Solve: $x^4 - 4x^3 + 5x^2 - 2x - 20 = 0.$

Begin by attempting to extract the square root.

$$\begin{array}{r|l}
 & x^4 - 4x^3 + 5x^2 - 2x - 20 \;\underline{|\, x^2 - 2x} \\
 & x^4 \\
2x^2 - 2x & -4x^3 + 5x^2 \\
 & \underline{-4x^3 + 4x^2} \\
 & \qquad\qquad x^2 - 2x - 20.
\end{array}$$

We see from the above that the equation may be written

$$(x^2 - 2x)^2 + x^2 - 2x - 20 = 0.$$

Put $x^2 - 2x = y$; the equation becomes

$$y^2 + y - 20 = 0.$$

Solving this, $\qquad y = -5$ or $+4$.

$$\therefore\ x^2 - 2x = -5, \text{ or } x^2 - 2x = 4.$$

Solving these two equations, we find for the four values of x,

$$1 + 2\sqrt{-1},\quad 1 - 2\sqrt{-1},\quad 1 + \sqrt{5},\quad 1 - \sqrt{5}.$$

(2) Solve: $\qquad x^2 + \dfrac{1}{x^2} + x + \dfrac{1}{x} = 4.$

Add 2 to both numbers,

$$x^2 + 2 + \frac{1}{x^2} + x + \frac{1}{x} = 6.$$

Put $x + \dfrac{1}{x} = y$; the equation becomes

$$y^2 + y = 6.$$

Solving this, $\qquad y = 2$ or -3.

$$\therefore\ x + \frac{1}{x} = 2, \text{ or } x + \frac{1}{x} = -3.$$

Solving these two equations, we find for the four values of x,

$$1,\quad 1,\quad \frac{-3 + \sqrt{5}}{2},\quad \frac{-3 - \sqrt{5}}{2}.$$

Exercise 27.

Solve:

1. $\sqrt{x+4}+\sqrt{2x-1}=6.$
2. $\sqrt{13x-1}-\sqrt{2x-1}=5.$
3. $\sqrt{x}+\sqrt{4+x}=3.$
4. $\sqrt{x^2-9}+21=x^2.$
5. $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}.$
6. $\sqrt{2x+1}-\sqrt{x+4}=\frac{\sqrt{x-3}}{3}.$
7. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$
8. $\sqrt{x+7}+\sqrt{x-5}+\sqrt{3x+9}=0.$
9. $\sqrt{x+5}+\sqrt{8-2x}+\sqrt{9-4x}=0.$
10. $\sqrt{7-x}+\sqrt{3x+10}+\sqrt{x+3}=0.$
11. $\sqrt{2x^2+3x+7}=2x^2+3x-5.$
12. $x^2-3x+2=6\sqrt{x^2-3x-3}.$
13. $6x^2-3x-2=\sqrt{2x^2-x}.$
14. $15x-3x^2-16=4\sqrt{x^2-5x+5}.$
15. $6x^2-21x+20=\sqrt{4x^2-14x+16}.$
16. $\sqrt{36x^2+12x+33}=41-8x-24x^2.$
17. $4x^4-12x^3+5x^2+6x-15=0.$
18. $x^4-10x^3+35x^2-50x+24=0.$
19. $x^4-4x^3-10x^2+28x-15=0.$
20. $18x^4+24x^3-7x^2-10x-88=0.$
21. $4x^4-12x^3+17x^2-12x-12=0.$
22. $\sqrt{x}+\sqrt{x+3}=\frac{6}{\sqrt{x+3}}.$

23. $6+\sqrt{x^2-1}=\dfrac{16}{\sqrt{x^2-1}}.$

24. $\dfrac{1}{\sqrt{x+1}}+\dfrac{1}{\sqrt{x-1}}=\dfrac{1}{\sqrt{x^2-1}}.$

25. $\dfrac{\sqrt{x+2}-\sqrt{x-2}}{\sqrt{x+2}+\sqrt{x-2}}=\dfrac{x}{2}.$

26. $\dfrac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2.$

27. $\dfrac{\sqrt{3x^2+4}-\sqrt{2x^2+1}}{\sqrt{3x^2+4}+\sqrt{2x^2+1}}=\dfrac{1}{7}.$

28. $\dfrac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7.$

29. $\dfrac{\sqrt{5x-4}+\sqrt{5-x}}{\sqrt{5x-4}-\sqrt{5-x}}=\dfrac{2\sqrt{x}+1}{2\sqrt{x}-1}.$

30. $\sqrt{(x+a)^2+2ab+b^2}+x+a=b.$

31. $\dfrac{\sqrt{3}}{\sqrt{2x-1}-\sqrt{x-2}}=\dfrac{1}{\sqrt{x-1}}.$

32. $\sqrt{\dfrac{x}{4}+3}+\sqrt{\dfrac{x}{4}-3}=\sqrt{\dfrac{2x}{3}}.$

33. $\sqrt{1+\dfrac{x}{a}}-\sqrt{1-\dfrac{a}{x}}=1.$

34. $\sqrt{x^2+a^2+3ax}+\sqrt{x^2+a^2-3ax}=\sqrt{2a^2+2b^2}.$

35. $4x^{\frac{1}{2}}-3(x^{\frac{1}{2}}+1)(x^{\frac{1}{2}}-2)=x^{\frac{1}{2}}(10-3x^{\frac{1}{2}}).$

36. $(x^{\frac{2}{3}}-2)(x^{\frac{4}{3}}-4)=x^{\frac{2}{3}}(x^{\frac{2}{3}}-1)^2-12.$

37. $3\sqrt{x^3+17}+\sqrt{x^3+1}+2\sqrt{5x^3+41}=0.$

38. $\frac{1}{2}-\frac{3}{x}=\sqrt{\frac{1}{4}-\frac{1}{x}\sqrt{9-\frac{36}{x}}}.$

39. $\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x.$

40. $\frac{1}{1+\sqrt{1-x}}+\frac{1}{1-\sqrt{1-x}}=\frac{2x}{9}.$

41. $\frac{\sqrt{ax+b}+\sqrt{ax}}{\sqrt{ax+b}-\sqrt{ax}}=\frac{1+\sqrt{ax-b}}{1-\sqrt{ax-b}}.$

42. $\frac{\sqrt{a-x}+\sqrt{b-x}}{\sqrt{a-x}-\sqrt{b-x}}=\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}}.$

43. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

44. $\left.\begin{aligned}x^2+y^2+x+y&=48\\ xy&=12\end{aligned}\right\}.$

45. $\left.\begin{aligned}x+y+\sqrt{x+y}&=a\\ x-y+\sqrt{x-y}&=b\end{aligned}\right\}.$

46. $\left.\begin{aligned}x^2+xy+y^2&=a^2\\ x+\sqrt{xy}+y&=b\end{aligned}\right\}.$

47. $\left.\begin{aligned}\frac{3\sqrt{x}+2\sqrt{y}}{4\sqrt{x}-2\sqrt{y}}&=6.\\ \frac{x^2+1}{16}&=\frac{y^2-64}{x^2}\end{aligned}\right\}.$

48. $\left.\begin{aligned}\sqrt{x}-\sqrt{y}&=x^{\frac{1}{2}}(\sqrt{x}+\sqrt{y})\\ (x+y)^2&=2(x-y)^2\end{aligned}\right\}.$

49. $\left.\begin{aligned}\sqrt{\frac{3x}{x+y}}+\sqrt{\frac{x+y}{3x}}&=2\\ x+y&=xy-54\end{aligned}\right\}.$

CHAPTER XII.

PROPERTIES OF QUADRATIC EQUATIONS.

151. Representing the roots of the quadratic equation $ax^2 + bx + c = 0$ by α and β, we have (§ 141),

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Adding, $\qquad \alpha + \beta = -\frac{b}{a};$

multiplying, $\qquad \alpha\beta = \frac{c}{a}.$

If we divide the equation $ax^2 + bx + c = 0$ through by a, we have the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$; this may be written $x^2 + px + q = 0$ where $p = \frac{b}{a}$, $q = \frac{c}{a}$.

It appears, then, that if any quadratic equation be made to assume the form $x^2 + px + q = 0$, the following relations hold between the coefficients and roots of the equation :

(1) The **sum** of the two roots is equal to the coefficient of x with its sign changed.

(2) The **product** of the two roots is equal to the constant term.

Thus the sum of the two roots of the equation $x^2 - 7x + 8 = 0$ is 7, and the product of the roots 8.

152.* The expressions $\alpha+\beta$, $\alpha\beta$, are examples of **symmetric functions** of the roots. Any expression which involves both roots, the two roots entering to similar powers and with similar coefficients, is a symmetric function of the roots.

From the relations $\alpha+\beta=-p$, $\alpha\beta=q$, the value of any symmetric function of the roots of a given quadratic may be found in terms of the coefficients.

Given that α and β are the roots of the quadratic $x^2-7x+8=0$, we may find the values of symmetric functions of the roots as follows:

(1) $\alpha^2+\beta^2$.

We have $\alpha+\beta=7,$

$\alpha\beta=8.$

Square the first, $\alpha^2+2\alpha\beta+\beta^2=49.$

Subtract, $2\alpha\beta=16,$

and we have $\alpha^2+\beta^2=33.$

(2) $\alpha^3+\beta^3$.

$\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3=343.$

$3\alpha\beta(\alpha+\beta)$ or $3\alpha^2\beta+3\alpha\beta^2=168.$

Subtract, $\alpha^3+\beta^3=175.$

(3) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$.

This is $\dfrac{\alpha^3+\beta^3}{\alpha\beta}$,

which is $\dfrac{175}{8}$.

153. Resolution into Factors. By § 151, if α and β are the roots of the equation $x^2+px+q=0$, the equation may be written

$$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

The left member is the product of $x-\alpha$ and $x-\beta$, so that the equation may be also written

$$(x-\alpha)(x-\beta)=0.$$

It appears, then, that the factors of the *quadratic expression* x^2+px+q are $x-\alpha$ and $x-\beta$, where α and β are the roots of the *quadratic equation* $x^2+px+q=0$.

The factors are real and different, real and alike, or imaginary, according as α and β are real and unequal, real and equal, or imaginary.

If $\beta=\alpha$, the equation becomes $(x-\alpha)(x-\alpha)=0$, or $(x-\alpha)^2=0$; if, then, the two roots of a quadratic equation be equal, the left member, when all the terms are transposed to that member, will be a perfect square as regards x.

If the equation be in the form $ax^2+bx+c=0$, the left member may be written $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$, or (§ 151)

$$a(x-\alpha)(x-\beta).$$

154. If the roots of a quadratic equation be given, we can readily form the equation.

Form the equation of which the roots are 3 and $-\frac{5}{2}$.

The equation is $\quad (x-3)\left(x+\frac{5}{2}\right)=0,$

or $\quad (x-3)(2x+5)=0,$

or $\quad 2x^2-x-15=0.$

155. Quadratic expressions may be factored by the principles of § 153.

(1) Resolve into two factors x^2-5x+3.

Write the equation $x^2-5x+3=0$.

The roots are found to be $\frac{5+\sqrt{13}}{2}$, $\frac{5-\sqrt{13}}{2}$.

The factors of x^2-5x+3 are

$$x-\frac{5+\sqrt{13}}{2} \text{ and } x-\frac{5-\sqrt{13}}{2}.$$

(2) Resolve into factors $3x^2-4x+5$.

Write the equation $3x^2-4x+5=0$.

The roots are found to be $\frac{2+\sqrt{-11}}{3}$, $\frac{2-\sqrt{-11}}{3}$.

Therefore the expression $3x^2-4x+5$ may be written (§ 153)

$$3\left(x-\frac{2+\sqrt{-11}}{3}\right)\left(x-\frac{2-\sqrt{-11}}{3}\right).$$

Exercise 28.

Form the equations of which the roots are:

1. $3, 2.$
2. $4, -5.$
3. $-6, -8.$
4. $\frac{2}{3}, \frac{1}{2}.$
5. $-\frac{1}{3}, -\frac{3}{4}.$
6. $a+3b, a-3b.$
7. $\frac{a+2b}{3}, \frac{2a+b}{3}.$
8. $2+\sqrt{3}, 2-\sqrt{3}.$
9. $-1+\sqrt{5}, -1-\sqrt{5}.$
10. $1+\sqrt{\frac{2}{3}}, 1-\sqrt{\frac{2}{3}}.$

Resolve into factors, real or imaginary:

11. $3x^2-15x-42.$
12. $9x^2-27x-70.$
13. $49x^2+49x+6.$
14. $169x^2-52x+4.$
15. $x^2-3x+4.$
16. $x^2+x+1.$
17. $4x^2-28x+49.$
18. $4x^2+12x+13.$

NOTE. The remainder of this chapter may be omitted if it is desired to abridge the course.

In examples 19–27, α and β are to be taken as the roots of the equation $x^2-7x+8=0$.

Find the values of:

19. $(\alpha-\beta)^2.$
20. $\alpha^2\beta+\alpha\beta^2.$
21. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}.$
22. $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}.$
23. $\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}.$
24. $\frac{\alpha^2+\beta^2}{\alpha+\beta}.$
25. $\frac{1}{\alpha^3}+\frac{1}{\beta^3}.$
26. $(\alpha^2-\beta^2)^2.$
27. $\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}.$

In examples 28–33 α and β are to be taken as the roots of the equation $x^2 + px + q = 0$. The results are to be found in terms of p and q.

Find the values of:

28. $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$.

29. $\alpha^2\beta + \alpha\beta^2$.

30. $\alpha^3 + \beta^3$.

31. $\alpha^3\beta + \alpha\beta^3$.

32. $\alpha^4 + \beta^4$.

33. $\dfrac{\alpha^2}{\beta^2} + \dfrac{\beta^2}{\alpha^2}$.

34. When will the roots of the equation $ax^2 + bx + c = 0$ be both positive? Both negative? One positive and one negative?

35. When will one root be the square of the other?

36. When will the sum of the reciprocals of the roots be unity?

37. Show that the roots of the equation

$$x^2 + 2(a+b)x + 2(a^2 + b^2) = 0$$

are imaginary if a and b are real and unequal.

38.* Show that the roots of the equation

$$x^2 + (x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b) = 0.$$

are ~~real~~ if a, b, and c are real.

39.* Show that the equations

$$ax^2 + bx + c = 0, \quad a'x + c' = 0,$$

will have a common root if $\dfrac{a}{a'^2} + \dfrac{c}{c'^2} = \dfrac{b}{a'c'}$.

40.* Show that the equations

$$ax^2 + bx + c = 0, \quad a'x^2 + b'x + c' = 0,$$

will have a common root if

$$(a'c - ac')^2 = (b'c - bc')(a'b - ab').$$

156.* The Roots in Special Cases. The values of the roots of the equation $ax^2 + bx + c = 0$ are (§ 141)

$$\frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \frac{-b-\sqrt{b^2-4ac}}{2a}. \qquad \textbf{A}$$

Multiplying both numerator and denominator of the first expression by $-b-\sqrt{b^2-4ac}$, and both numerator and denominator of the second expression by $-b+\sqrt{b^2-4ac}$, we obtain these new forms for the values of the roots:

$$\frac{2c}{-b-\sqrt{b^2-4ac}}, \quad \frac{2c}{-b+\sqrt{b^2-4ac}}. \qquad \textbf{B}$$

We proceed to consider the following special cases:

I. Suppose a to be very small compared with b and c. In this case b^2-4ac differs but little from b^2, and its square root but little from b. The denominator of the first root in **B** will be very nearly $-2b$, and the root itself very nearly $-\frac{c}{b}$; the denominator of the second root in **B** will be very small, and the root itself numerically very large.

The smaller a is, the larger will the second root be, and the less will the first root differ from $-\frac{c}{b}$.

The first root may be found approximately by neglecting the x^2 term and solving the simple equation $bx+c=0$. In fact, the quadratic equation itself approximates to the form $0x^2+bx+c=0$.

II. Suppose both a and b to be very small compared with c. In this case the first root, which differs but little from $-\frac{c}{b}$, also becomes very large, so that both roots are very large.

The smaller a and b are, the larger will the roots be.

The quadratic equation in this case approximates to the form $0x^2 + 0x + c = 0$.

III. Suppose $c = 0$ while a and b are not zero. In this case the first root in **A** becomes zero, the second root becomes $-\frac{b}{a}$.

The quadratic equation becomes

$$ax^2 + bx = 0 \text{ or } x(ax + b) = 0;$$

one root is 0, the other, obtained by solving the equation $ax + b = 0$, is $-\frac{b}{a}$ (§ 140); these are the values just found.

IV. Suppose $b = 0$ and $c = 0$ while a is not zero. In this case both roots in **A** become zero.

The equation reduces to $ax^2 = 0$, of which both roots are zero (§ 140).

V. Suppose $b = 0$ while a and c are not zero. In this case the two roots become $+\sqrt{-\frac{c}{a}}$ and $-\sqrt{-\frac{c}{a}}$.

The equation becomes the pure quadratic $ax^2 + c = 0$. Solving this, we obtain for x the values just found.

157.* Collecting results, we have the following:

I. a very small compared with b and c; one root very large.

II. a and b both very small compared with c; both roots very large.

III. $c = 0$, a and b not zero; one root zero.

IV. $b = 0$, $c = 0$, a not zero; both roots zero.

V. $b = 0$, a and c not zero; a pure quadratic; roots numerically equal but opposite in sign.

158.* Variable Coefficients. When the coefficients of an equation involve an undetermined number, the character of the roots may depend on the value given to the unknown number.

For what values of m will the equation

$$2mx^2 + (5m+2)x + (4m+1) = 0$$

have its roots real and equal, real and unequal, imaginary?

We find
$$\begin{aligned} b^2 - 4ac &= (5m+2)^2 - 8m(4m+1) \\ &= 4 + 12m - 7m^2 \\ &= (2-m)(2+7m). \end{aligned}$$

Roots *equal.* In this case $b^2 - 4ac$ is to be zero.

We must have either

$$2 - m = 0, \text{ or } 2 + 7m = 0.$$

$$\therefore m = 2, \text{ or } \quad m = -\frac{2}{7}.$$

Roots *real and unequal.* In this case $b^2 - 4ac$ is to be positive.

The factors $2-m$, $2+7m$, are to be both positive or both negative.

If m lies between 2 and $-\frac{2}{7}$, both factors are positive; both factors cannot be negative.

Roots *imaginary.* In this case $b^2 - 4ac$ is to be negative.

Of the two factors $2-m$, $2+7m$, one is to be positive and the other negative.

If m is algebraically greater than 2, $2-m$ is negative and $2+7m$ positive; if m is algebraically less than $-\frac{2}{7}$, $2+7m$ is negative and $2-m$ positive.

159.* By a method similar to that of the last section, we can often obtain the maximum or minimum value of a quadratic expression for real values of x.

(1) Find the maximum or minimum value of $1 + x - x^2$ for real values of x.

Let $\quad 1 + x - x^2 = m;$

then $\quad x^2 - x = 1 - m.$

Solving, $$x = \frac{1 \pm \sqrt{5 - 4m}}{2}.$$

Since x is real, we must have

$$5 > 4m \text{ or } 5 = 4m.$$

$$\therefore\ 4m \text{ is not greater than } 5,$$

$$m \text{ is not greater than } \frac{5}{4}.$$

The maximum value of $1 + x - x^2$ is $\frac{5}{4}$; for this value $x = \frac{1}{2}$.

(2) Find the maximum or minimum value of $x^2 + 3x + 4$ for real values of x.

Let $$x^2 + 3x + 4 = m;$$

then $$x^2 + 3x = m - 4.$$

Solving, $$x = \frac{-3 \pm \sqrt{4m - 7}}{2}.$$

Since x is real, we must have

$$4m > 7 \text{ or } 4m = 7.$$

$$\therefore\ 4m \text{ is not less than } 7,$$

$$m \text{ is not less than } \frac{7}{4}.$$

The minimum value of $x^2 + 3x + 4$ is $\frac{7}{4}$; for this value $x = -\frac{3}{2}$.

NOTE. Instead of solving for x, we might have used the condition for real roots, viz., $b^2 - 4ac$ greater than or equal to zero.

160.* The existence of a maximum or minimum value may also be shown as follows:

Take the first expression of the last article,

$$1 + x - x^2.$$

This is $$\frac{5}{4} - \left(\frac{1}{4} - x + x^2\right),$$

or $$\frac{5}{4} - \left(x - \frac{1}{2}\right)^2.$$

$\left(x - \frac{1}{2}\right)^2$ is positive for all real values of x; its *least* value is zero, and in this case the given expression has its *greatest* value, $\frac{5}{4}$.

Similarly for any other expression.

Exercise 29.*

For what values of m are the two roots of each of the following equations (1) equal, (2) real and unequal, (3) imaginary?

1. $(3m+1)x^2+2(m+1)x+m=0.$

2. $(m-2)x^2+(m-5)x+2m-5=0.$

3. $2mx^2+x^2-6mx-6x+6m+1=0.$

4. $mx^2+2x^2+2m-3mx+9x-10=0.$

5. $6mx^2+8mx+2m=2x-x^2-1.$

Find the maximum or minimum value of each of the following expressions, and determine which:

6. $x^2-6x+13.$

7. $4x^2-12x+16.$

8. $3+12x-9x^2.$

9. $x^2+8x+20.$

10. $4x^2-12x+25.$

11. $25x^2-40x-16.$

12. $\dfrac{x-6}{x^2}.$

13. $\dfrac{(x+12)(x-3)}{x^2}.$

14. $\dfrac{4x}{(x+2)^2}.$

15. $\dfrac{x^2-x-1}{x^2-x+1}.$

16. $\dfrac{x^2+2x-3}{x^2-2x+3}.$

17. $\dfrac{1}{2+x}-\dfrac{1}{2-x}.$

18. $\dfrac{x^2+3x+5}{x^2+1}.$

19. $\dfrac{(x+1)^2}{x^2-x+1}.$

20. $\dfrac{2x^2-2x+5}{x^2-2x+3}.$

21. Divide a line $2a$ inches long into two parts such that the rectangle of these parts shall be the greatest possible.

22. Divide a line 20 inches long into two parts such that the hypotenuse of the right triangle of which the two parts are the legs shall be the least possible.

23. Divide $2a$ into two parts such that the sum of their square roots shall be a maximum.

24. Find the greatest rectangle that can be inscribed in a given triangle.

25. Find the greatest rectangle that can be inscribed in a given circle.

26. Find the rectangle of greatest perimeter that can be inscribed in a given circle.

CHAPTER XIII.

SURDS AND IMAGINARIES.

161. Quadratic Surds. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.*

For every quadratic surd, when simplified, will have under the radical sign one or more factors raised only to the first power; and two surds which are *dissimilar* cannot have *all* these factors alike.

162. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For if $\sqrt{a} \pm \sqrt{b}$ could equal a rational number c, we should have, by squaring, and transposing,

$$\pm 2\sqrt{ab} = c^2 - a - b.$$

Now, as the right side of this equation is rational, the left side would be rational; but, by § 161, $\sqrt{ab}$ cannot be rational. Therefore $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

In like manner, it may be shown that $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

163. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For, if $\sqrt{a}$ could equal $c + \sqrt{b}$, we should have, squaring, and transposing,

$$2c\sqrt{b} = a - b - c^2.$$

That is, a surd equal to a rational number, which is impossible.

164. *If* $a+\sqrt{b}=x+\sqrt{y}$, *then* a *will equal* x, *and* b *will equal* y.

For, transposing, $\sqrt{b}-\sqrt{y}=x-a$; and if b were not equal to y, the difference of two unequal surds would be rational, which by § 62 is impossible.

$$\therefore\ b=y \text{ and } a=x.$$

In like manner, if $a-\sqrt{b}=x-\sqrt{y}$, a will equal x, and b will equal y.

Expressions of the form $a+\sqrt{b}$, where $\sqrt{b}$ is a surd, are called **binomial surds.**

165. Square Root of a Binomial Surd.

Let $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.

Squaring, $a+\sqrt{b}=x+2\sqrt{xy}+y$.

$\therefore\ x+y=a$, and $2\sqrt{xy}=\sqrt{b}$. (§ 164)

From these two equations the values of x and y may be found.

This method may be shortened by observing that, since $\sqrt{b}=2\sqrt{xy}$, we have

$$a-\sqrt{b}=x-2\sqrt{xy}+y.$$

By taking the root, $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

$$\therefore\ (\sqrt{a+\sqrt{b}})(\sqrt{a-\sqrt{b}})=(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}).$$

$$\therefore\ \sqrt{a^2-b}=x-y.$$

And, as $a=x+y$,

the values of x and y may be found by addition and subtraction.

(1) Extract the square root of $7+4\sqrt{3}$.

Let $\sqrt{x}+\sqrt{y}=\sqrt{7+4\sqrt{3}}$.

Then $\sqrt{x}-\sqrt{y}=\sqrt{7-4\sqrt{3}}$.

Multiplying, $x-y=\sqrt{49-48}$,

$\therefore\ x-y=1$.

But $x+y=7$,

$\therefore\ x=4$, and $y=3$.

$$\therefore \sqrt{x} + \sqrt{y} = 2 + \sqrt{3}.$$
$$\therefore \sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}.$$

A root may often be obtained by inspection. For this purpose, write the given expression in the form $a + 2\sqrt{b}$, and determine what two numbers have their sum equal to a, and their product equal to b.

(2) Find by inspection the square root of $18 + 2\sqrt{77}$.

It is required to find two numbers whose sum is 18 and whose product is 77; these are evidently 11 and 7.

Then $\qquad 18 + 2\sqrt{77} = 11 + 7 + 2\sqrt{11 \times 7},$
$\qquad\qquad = (\sqrt{11} + \sqrt{7})^2.$

That is, $\qquad \sqrt{11} + \sqrt{7} =$ square root of $18 + 2\sqrt{77}$.

(3) Find by inspection the square root of $75 - 12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore, $75 - 12\sqrt{21}$ must be put in the form

$$75 - 2\sqrt{756}.$$

The two numbers whose sum is 75 and whose product is 756 are 63 and 12.

Then $\qquad 75 - 2\sqrt{756} = 63 + 12 - 2\sqrt{63 \times 12},$
$\qquad\qquad = (\sqrt{63} - \sqrt{12})^2.$

That is, $\qquad \sqrt{63} - \sqrt{12} =$ square root of $75 - 12\sqrt{21}$;

or, $\qquad 3\sqrt{7} - 2\sqrt{3} =$ square root of $75 - 12\sqrt{21}$.

Exercise 30.

Extract the square roots of:

1. $14 + 6\sqrt{5}$.
2. $17 + 4\sqrt{15}$.
3. $10 + 2\sqrt{21}$.
4. $16 + 2\sqrt{55}$.
5. $9 - 2\sqrt{14}$.
6. $20 - 8\sqrt{6}$.
7. $9 - 6\sqrt{2}$.
8. $94 - 42\sqrt{5}$.
9. $13 - 2\sqrt{30}$.
10. $11 - 6\sqrt{2}$.
11. $14 - 4\sqrt{6}$.
12. $38 - 12\sqrt{10}$.
13. $103 - 12\sqrt{11}$.
14. $57 - 12\sqrt{15}$.
15. $3\frac{1}{2} - \sqrt{10}$.
16. $2a + 2\sqrt{a^2 - b^2}$.
17. $a^2 - 2b\sqrt{a^2 - b^2}$.
18. $87 - 12\sqrt{42}$.
19. $(a + b)^2 - 4(a - b)\sqrt{ab}$.

IMAGINARY EXPRESSIONS.

156. An **imaginary expression** is any expression which involves the indicated even root of a negative number.

It will be shown hereafter that *any* indicated even root of a negative number may be made to assume a form which involves only indicated *square roots* of negative numbers. In considering imaginary expressions, we accordingly need consider only expressions which involve the indicated square roots of negative numbers.

Imaginary expressions are also called **imaginary numbers** and **complex numbers**.

157. **Imaginary Square Roots.** If a and b are both positive, we have § [illegible]

$$\text{I. } \sqrt{ab} = \sqrt{a}\sqrt{b}. \qquad \text{II. } (\sqrt{a})^2 = a.$$

If one of the two numbers a and b is positive and the other negative, law I is *assumed* still to apply; we have accordingly

$$\sqrt{-4} = \sqrt{4(-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1};$$

$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{5}\sqrt{-1};$$

$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a}\sqrt{-1};$$

and so on.

[illegible] square root can [illegible] where a is a real number.

158. [illegible] cannot apply,

[illegible]

II. gives

$$(+\sqrt{-a})(+\sqrt{-a}) = (+\sqrt{-a})^2 = -a.$$

We therefore *assume* that II. holds true; hence I. does not hold true. This assumption gives us:

$$\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1;$$

$$\begin{aligned}\sqrt{-a} \times \sqrt{-b} &= \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1}\\ &= \sqrt{a}\sqrt{b}(\sqrt{-1})^2\\ &= \sqrt{ab}(-1)\\ &= -\sqrt{ab}.\end{aligned}$$

The law $\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1$ is very important.

Observe that the law $\sqrt{a}\sqrt{b} = \sqrt{ab}$ holds true unless *both* a and b are negative.

169. It will be useful to form the successive powers of $\sqrt{-1}$.

$$\begin{aligned}&(\sqrt{-1})^2 = -1;\\ &(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} \quad = (-1)\sqrt{-1} = -\sqrt{-1};\\ &(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1;\\ &(\sqrt{-1})^5 = (\sqrt{-1})^4\sqrt{-1} \quad = (+1)\sqrt{-1} = +\sqrt{-1};\end{aligned}$$

and so on. It appears that the successive powers of $\sqrt{-1}$ form the repeating series $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$, and so on.

170. An imaginary expression will generally consist of two parts: a real part and an imaginary part. Thus, the roots of the quadratic equation $x^2 - 6x + 13 = 0$ are $3 + \sqrt{-4}$, $3 - \sqrt{-4}$; that is, $3 + 2\sqrt{-1}$, $3 - 2\sqrt{-1}$; each of these imaginary expressions consists of a real part and an imaginary part.

171. Every imaginary expression may be made to assume the form $a+b\sqrt{-1}$, where a and b are real numbers, and may be integers, fractions, or surds.

If $b=0$, the expression consists of only the real part a, and is therefore real.

If $a=0$, the expression consists of only the imaginary part $b\sqrt{-1}$, and is a **pure imaginary.**

172. The form $a+b\sqrt{-1}$ is the **typical form** of imaginary expressions.

Reduce to the typical form $6+\sqrt{-8}$.

This may be written $6+\sqrt{8}\sqrt{-1}$, or $6+2\sqrt{2}\sqrt{-1}$; here $a=6$, and $b=2\sqrt{2}$.

173. The **sum** of two imaginary expressions is generally an imaginary expression.

$$\begin{array}{ll} \text{Add} & a+b\sqrt{-1}, \\ \text{and} & \underline{c+d\sqrt{-1}.} \\ \text{The sum is} & (a+c)+(b+d)\sqrt{-1}. \end{array}$$

This is an imaginary expression unless $b+d=0$; in which case the expression is real.

174. The **product** of two imaginary expressions is generally an imaginary expression.

$$\begin{array}{ll} \text{Multiply} & a+b\sqrt{-1}, \\ \text{by} & \underline{c+d\sqrt{-1}.} \\ & ac+bc\sqrt{-1} \\ & \underline{\quad +ad\sqrt{-1}-bd} \\ \text{The product is} & (ac-bd)+(bc+ad)\sqrt{-1}, \end{array}$$

an imaginary expression unless $bc+ad=0$.

175. The **quotient** of two imaginary expressions is generally an imaginary expression.

Divide $a+b\sqrt{-1}$ by $c+d\sqrt{-1}$.

The quotient is $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}$.

Multiply both numerator and denominator by $c-d\sqrt{-1}$.

Then
$$\frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{(c+d\sqrt{-1})(c-d\sqrt{-1})}$$
$$=\frac{(ac+bd)+(bc-ad)\sqrt{-1}}{c^2+d^2},$$
$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\sqrt{-1}.$$

This is an imaginary expression in the typical form; if $bc-ad=0$, the quotient is real.

176. Two expressions of the form $a+b\sqrt{-1}$, $a-b\sqrt{-1}$ are called **conjugate imaginaries.**

Add $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$.
The sum is $2a$.

$$\begin{array}{lrl}
\text{Multiply} & a+b\sqrt{-1}, & \\
\text{by} & a-b\sqrt{-1}. & \\
\hline
 & a^2+ab\sqrt{-1} & \\
 & -ab\sqrt{-1} & +b^2 \\
\hline
\text{The product is,} & a^2 & +b^2.
\end{array}$$

From the above it appears that the *sum* and *product* of two conjugate imaginaries are both *real*.

The roots of a quadratic equation, if imaginary, are conjugate imaginaries (§ 141).

164. *If a*
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Examples :

(1) The roots of the equation $x^2-6x+13=0$ are $3+2\sqrt{-1}$ and $3-2\sqrt{-1}$. The expression $x^2-6x+13$ may be written $(x-3)^2+4$, which is positive for all real values of x.

(2) The roots of the equation $12x-13-4x^2=0$ are $\frac{3+2\sqrt{-1}}{2}$, $\frac{3-2\sqrt{-1}}{2}$. The expression $12x-13-4x^2$ may be written

$$-(4x^2-12x+9+4), \text{ or } -[(2x-3)^2+4],$$

which is negative for all real values of x.

The above expressions can never become zero; they accordingly have either a *minimum* value below which they cannot fall, or a *maximum* value above which they cannot rise (§ 159).

Exercise 31.

1. Multiply :

$\sqrt{-8}$ by $\sqrt{-2}$; $2\sqrt{-3}$ by $4\sqrt{-27}$; $3\sqrt{-5}$ by $\frac{3}{\sqrt{27}}$.

2. Divide :

$\sqrt{7}$ by $\sqrt{-3}$; $\sqrt{-8}$ by $\sqrt{-2}$; $3\sqrt{-6}$ by $\sqrt{2}\sqrt{-3}$.

3. Reduce to the typical form :

$$4+\sqrt{-81};\ 5+2\sqrt{-6};\ (3+\sqrt{-27})^2.$$

Multiply :

4. $4+\sqrt{-3}$ by $4-\sqrt{-3}$.

5. $\sqrt{3}-2\sqrt{-2}$ by $\sqrt{3}+2\sqrt{-2}$.

6. $7+\sqrt{-27}$ by $4+\sqrt{-3}$.

7. $5+2\sqrt{-8}$ by $3-5\sqrt{-2}$.

8. $2\sqrt{3}-6\sqrt{-5}$ by $4\sqrt{3}-\sqrt{-5}$.

9. $\sqrt{a}+b\sqrt{-c}$ by $\sqrt{c}+a\sqrt{-b}$.

22. Divide a line 20 inches long into two parts such that the hypotenuse of the right triangle of which the two parts are the legs shall be the least possible.

23. Divide $2a$ into two parts such that the sum of their square roots shall be a maximum.

24. Find the greatest rectangle that can be inscribed in a given triangle.

25. Find the greatest rectangle that can be inscribed in a given circle.

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CHAPTER XIII.

SURDS AND IMAGINARIES.

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162. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For if $\sqrt{a} \pm \sqrt{b}$ could equal a rational number c, we should have, by squaring, and transposing,

$$\pm 2\sqrt{ab} = c^2 - a - b.$$

Now, as the right side of this equation is rational, the left side would be rational; but, by § 161, $\sqrt{ab}$ cannot be rational. Therefore $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

In like manner, it may be shown that $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

163. *A quadratic surd cannot equal the sum of a rational number and a surd.*

For, if $\sqrt{a}$ could equal $c + \sqrt{b}$, we should have, squaring, and transposing,

$$2c\sqrt{b} = a - b - c^2.$$

That is, a surd equal to a rational number, which is impossible.

(1) If a and b are positive, show that $a^3+b^3>a^2b+ab^2$.

We shall have $$a^3+b^3>a^2b+ab^2,$$
if (dividing each side by $a+b$),
$$a^2-ab+b^2>ab,$$
if $$a^2+b^2>2ab.$$
But this is true (§ 83). $\therefore a^3+b^3>a^2b+ab^2$.

(2) Show that $a^2+b^2+c^2>ab+ac+bc$.

Now,
$$a^2+b^2>2ab,$$
$$a^2+c^2>2ac, \qquad (§ 183)$$
$$b^2+c^2>2bc.$$
Adding, $$2a^2+2b^2+2c^2>2ab+2ac+2bc,$$
$$\therefore a^2+b^2+c^2>ab+ac+bc.$$

Exercise 32.

Show that, the letters being unequal and positive:

1. $a^2+3b^2>2b(a+b)$.
2. $a^3b+ab^3>2a^2b^2$.
3. $(a^2+b^2)(a^4+b^4)>(a^3+b^3)^2$.
4. $a^2b+a^2c+ab^2+b^2c+ac^2+bc^2>6abc$.
5. The sum of any fraction and its reciprocal >2.
6. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, $xy>ac+bd$, or $ad+bc$.
7. $ab+ac+bc<(a+b-c)^2+(a+c-b)^2+(b+c-a)^2$.
8. Which is the greater, $(a^2+b^2)(c^2+d^2)$ or $(ac+bd)^2$?
9. Which is the greater, a^4-b^4 or $4a^3(a-b)$ when $a>b$?
10. Which is the greater, $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}$ or $\sqrt{a}+\sqrt{b}$?
11. Which is the greater, $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$?
12. Which is the greater, $\frac{a}{b^2}+\frac{b}{a^2}$ or $\frac{1}{b}+\frac{1}{a}$?

CHAPTER XV.

When

RATIO, PROPORTION, AND VARIATION.

185. Ratio of Numbers. The relative magnitude of two numbers is called their **ratio,** ~~and is~~ expressed by the indicated quotient of the first by the second. Thus the ratio of a to b is $\frac{a}{b}$, or $a \div b$, or $a : b$; the quotient is generally written in the last form when it is intended to express a ratio.

The first term of a ratio is called the **antecedent,** and the second term the **consequent.** When the antecedent is equal to the consequent, the ratio is called a ratio of *equality;* when the antecedent is greater than the consequent, the ratio is called a ratio of *greater inequality;* when less, a ratio of *less inequality.*

When the antecedent and consequent are interchanged, the resulting ratio is called the *inverse* of the given ratio.

Thus, the ratio 3 : 6 is the *inverse* of the ratio 6 : 3.

186. A ratio will not be altered if both its terms be multiplied by the same number.

For the ratio $a : b$ is represented by $\frac{a}{b}$, the ratio $ma : mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb} = \frac{a}{b}$, we have $ma : mb = a : b$.

A ratio will be altered if different multipliers of its terms be taken; and will be increased or diminished according as the multiplier of the antecedent is greater than or less than that of the consequent.

If	$m > n,$	If	$m < n,$
	$ma > na,$		$ma < na,$
and	$\frac{ma}{nb} > \frac{na}{nb};$	and	$\frac{ma}{nb} < \frac{na}{nb};$
but	$\frac{na}{nb} = \frac{a}{b}.$	but	$\frac{na}{nb} = \frac{a}{b}.$
$\therefore$	$\frac{ma}{nb} > \frac{a}{b},$	$\therefore$	$\frac{ma}{nb} < \frac{a}{b},$
or	$ma : nb > a : b.$	or	$ma : nb < a : b.$

187. Ratios are *compounded* by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

The ratio compounded of $a : b$ and $a : b$ is the *duplicate* ratio $a^2 : b^2$; the ratio compounded of $a : b$, $a : b$, and $a : b$ is the *triplicate* ratio $a^3 : b^3$; and so on.

188. Ratios are *compared* by comparing the fractions that represent them.

Thus, $\qquad a : b >$ or $< c : d$

according as $\qquad \frac{a}{b} >$ or $< \frac{c}{d},$

as $\qquad \frac{ad}{bd} >$ or $< \frac{bc}{bd},$

as $\qquad ad >$ or $< bc.$

189. Proportion of Numbers. Four numbers, a, b, c, d, are said to be in **proportion** when the ratio $a : b$ is equal to the ratio $c : d$.

We then write $a : b = c : d$, and read this either, the ratio of a to b equals the ratio of c to d, or a is to b as c is to d.

A proportion is also written $a : b :: c : d$.

The four numbers a, b, c, d are called *proportionals*; a and d are called the *extremes*, b and c the *means*.

190. When four numbers are in proportion, the product of the extremes is equal to the product of the means.

For, if $$a : b = c : d,$$
then $$\frac{a}{b} = \frac{c}{d}.$$
Multiplying by bd, $$ad = bc.$$

The equation $ad = bc$ gives

$$a = \frac{bc}{d}, \quad b = \frac{ad}{c};$$

so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean. If three terms of a proportion are given, it appears from the above that the fourth term can have one, and but one, value.

191. If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.

For, if $$ad = bc,$$
then, dividing by bd, $$\frac{ad}{bd} = \frac{bc}{bd},$$
or $$\frac{a}{b} = \frac{c}{d}.$$
$$\therefore a : b = c : d.$$

192. Transformations of a Proportion. If four numbers, a, b, c, d, be in proportion, they will be in proportion by:

I. Inversion: b will be to a as d is to c.

For, if $$a : b = c : d,$$
then $$\frac{a}{b} = \frac{c}{d},$$

and $$1 + \frac{a}{b} = 1 + \frac{c}{d},$$

or $$\frac{b}{a} = \frac{d}{c}.$$

$$\therefore b : a = d : c.$$

II. Composition: $a + b$ will be to b as $c + d$ is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore a + b : b = c + d : d.$$

III. Division: $a - b$ will be to b as $c - d$ is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

$$\therefore a - b : b = c - d : d.$$

IV. Composition and Division: $a + b$ will be to $a - b$ as $c + d$ is to $c - d$.

For, from II., $$\frac{a+b}{b} = \frac{c+d}{d},$$

and from III., $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Dividing, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore a + b : a - b = c + d : c - d.$$

V. Alternation: a will be to c as b is to d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying by $\frac{b}{c}$, $$\frac{ab}{bc} = \frac{bc}{cd},$$

or $$\frac{a}{c} = \frac{b}{d}.$$

$$\therefore a : c = b : d.$$

193. In a series of **equal ratios,** the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

For, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h},$$

r may be put for each of these ratios.

Then $$\frac{a}{b} = r, \ \frac{c}{d} = r, \ \frac{e}{f} = r, \ \frac{g}{h} = r.$$

$$\therefore a = br, \ c = dr, \ e = fr, \ g = hr.$$

$$\therefore a + c + e + g = (b + d + f + h)r.$$

$$\therefore \frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$$

$$\therefore a + c + e + g : b + d + f + h = a : b.$$

In like manner it may be shown that

$$ma + nc + pe + qg : mb + nd + pf + qh = a : b.$$

194. If a, b, c, d be in **continued proportion,** that is, if $a : b = b : c = c : d$, then will $a : c = a^2 : b^2$ and $a : d = a^3 : b^3$.

For, $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Hence, $$\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b},$$

or $$\frac{a}{c} = \frac{a^2}{b^2}.$$

$$\therefore a : c = a^2 : b^2.$$

Also $$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b},$$

or
$$\frac{a}{d}=\frac{a^3}{b^3}.$$
$$\therefore a:d=a^3:b^3.$$

195. If a, b, c are proportionals, so that $a:b=b:c$, then b is called a **mean proportional** between a and c, and c is called a **third proportional** to a and b.

If $\qquad a:b=b:c$, then $b=\sqrt{ac}$.

For, if
$$a:b=b:c,$$
then
$$\frac{a}{b}=\frac{b}{c},$$
and
$$b^2=ac.$$
$$\therefore b=\sqrt{ac}.$$

196. The products of the corresponding terms of two or more proportions are in proportion.

For, if
$$a:b=c:d,$$
$$e:f=g:h,$$
$$k:l=m:n,$$
then
$$\frac{a}{b}=\frac{c}{d},\ \frac{e}{f}=\frac{g}{h},\ \frac{k}{l}=\frac{m}{n}$$

Taking the product of the left members, and also of the right members of these equations,
$$\frac{aek}{bfl}=\frac{cgm}{dhn}.$$
$$\therefore aek:bfl=cgm:dhn.$$

197. Like powers, or like roots, of the terms of a proportion are in proportion.

For, if
$$a:b=c:d,$$
then
$$\frac{a}{b}=\frac{c}{d}.$$

Raising both sides to the nth power,
$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$
$$\therefore a^n:b^n=c^n:d^n.$$

Extracting the *nth* root,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$$

$$\therefore a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

198. If two numbers be increased or diminished by like parts of each, the results will be in the same ratio as the numbers themselves.

For $$\frac{a}{b}=\frac{\left(1\pm\frac{m}{n}\right)a}{\left(1\pm\frac{m}{n}\right)b}=\frac{a\pm\frac{m}{n}a}{b\pm\frac{m}{n}b}.$$

$$\therefore a : b = a\pm\frac{m}{n}a : b\pm\frac{m}{n}b.$$

199. The laws that have been established for ratios should be remembered when ratios are expressed in fractional form.

(1) Solve: $$\frac{x^2+x+1}{x^2-x-1}=\frac{x^2-x+2}{x^2+x-2}.$$

By composition and division,

$$\frac{2x^2}{2(x+1)}=\frac{2x^2}{-2(x-2)},$$

and this equation is satisfied, when $x = 0$;

or, dividing by $\frac{2x^2}{2}$, when $\frac{1}{x+1}=\frac{1}{2-x}$;

that is, when $x=\frac{1}{2}$.

(2) If $a : b = c : d$, show that

$$a^2+ab : b^2-ab = c^2+cd : d^2-cd.$$

If $$\frac{a}{b}=\frac{c}{d},$$

then $$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$

and
$$\frac{a}{-b}=\frac{c}{-d}.$$

$$\therefore \frac{a}{-b}\times\frac{a+b}{a-b}=\frac{c}{-d}\times\frac{c+d}{c-d};$$

that is,
$$\frac{a^2+ab}{b^2-ab}=\frac{c^2+cd}{d^2-cd},$$

or
$$a^2+ab : b^2-ab = c^2+cd : d^2-cd.$$

(3) If $a : b = c : d$, and a is the *greatest term*, show that $a + d$ is greater than $b + c$.

Since
$$\frac{a}{b}=\frac{c}{d}, \text{ and } a > c,$$
$$\therefore b > d.$$

Also,
$$\frac{a-b}{b}=\frac{c-d}{d},$$

and
$$b > d.$$
$$\therefore a-b > c-d.$$

Adding
$$b+d=b+d,$$

we have
$$a+d > b+c.$$

Exercise 33.

1. Write down the ratio compounded of $3:5$ and $8:7$. Which of these ratios is increased, and which is diminished by the composition?

2. Compound the duplicate ratio of $4:15$ with the triplicate of $5:2$.

3. Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater inequality or a ratio of less inequality.

4. Arrange in order of magnitude the ratios $3:4$, $23:25$, $10:11$.

5. If $a > b$, which is the greater ratio,
$$a+b : a-b \text{ or } a^2+b^2 : a^2-b^2?$$

Find the ratios compounded of:

6. $3:5,\ 10:21,\ 14:15.$ **7.** $7:9,\ 102:105,\ 15:17.$

8. $a^2-x^2:a^2+3ax+2x^2$ and $a+x:a-x.$

9. $x^2-4:2x^2-5x+3$ and $x-1:x-2.$

10. Prove that a ratio of greater inequality is diminished, and a ratio of less inequality increased, by adding the same number to both its terms.

11. Prove that a ratio of greater inequality is increased, and a ratio of less inequality diminished, by subtracting the same number from both its terms.

12. Show that the ratio $a:b$ is the duplicate of the ratio $a+c:b+c$, if $c^2=ab$.

13. Two numbers are in the ratio $2:5$, and if 6 be added to each, they are in the ratio $4:7$. Find the numbers.

14. What must be added to each of the terms of the ratio $m:n$, that it may become equal to the ratio $p:q$?

15. If x and y be such that, when they are added to the antecedent and consequent respectively of the ratio $a:b$, its value is unaltered, show that $x:y=a:b$.

Find x from the proportions:

16. $27:90=45:x.$

17. $11\frac{1}{4}:4\frac{1}{2}=3\frac{3}{4}:x.$

18. $\dfrac{3a}{5b}:\dfrac{12a}{7c}=\dfrac{14c}{15b}:x.$

Find a third proportional to:

19. $\dfrac{10}{27}$ and $\dfrac{5}{12}.$ **20.** $\dfrac{a^2-b^2}{c}$ and $\dfrac{a-b}{c}.$

Find a mean proportional between:

21. 3 and $16\frac{1}{3}$. **22.** $\frac{(m-5)^2}{m+5}$ and $\frac{(m+5)^2}{m-5}$.

If $a:b=c:d$, prove that:

23. $2a+b:b=2c+d:d$. **24.** $3a-b:a=3c-d:c$.

25. $4a+3b:4a-3b=4c+3d:4c-3d$.

26. $2a^3+3b^3:2a^3-3b^3=2c^3+3d^3:2c^3-3d^3$.

If $a:b=b:c$, prove that:

27. $a^2+ab:b^2+bc::a:c$. **28.** $a:c::(a+b)^2:(b+c)^2$.

29. If $a:b=b:c$, and a is the greatest of the three numbers, show that $a+c>2b$.

30. If $\frac{x-y}{l}=\frac{y-z}{m}=\frac{z-x}{n}$, and x, y, z be unequal, show that $l+m+n=0$.

Find x from the proportions:

31. $x+1:x-1=x+2:x-2$.

32. $x+a:2x-b=3x+b:4x-a$.

33. $x^2-4x+2:x^2-2x-1=x^2-4x:x^2-2x-2$.

34. $3+x:4+x=9+x:13+x$.

35. $a+x:b+x=c+x:d+x$.

36. If $a:b=c:d$, show that

$$a^2+b^2:\frac{a^3}{a+b}=c^2+d^2:\frac{c^3}{c+d}.$$

37. When a, b, c, d are proportional and all unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$ shall be proportionals.

200. Ratio of Quantities. To *measure* a quantity of any kind is to find out how many times it contains another *known* quantity of the *same kind*, called the **unit of measure.**

The number which expresses the number of times that a quantity contains the unit of measure is called the **numerical measure** of that quantity.

Thus, if a line contains the linear unit of measure, one yard, 5 times, the *measure* of the length of the line is 5 yards, and the *numerical measure* of the line is 5.

201. Commensurable Quantities. If two quantities of the *same kind* are so related that a unit of measure can be found which is contained in each of the quantities an integral number of times, this unit of measure is a *common measure* of the two quantities, and the two quantities are said to be **commensurable.**

If two commensurable quantities be measured by the same unit, their **ratio** is simply the ratio of their numerical measures.

Thus, $\frac{1}{6}$ of a foot is a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet, being contained in the first 15 times and in the second 22 times.

The ratio of $2\frac{1}{2}$ feet to $3\frac{2}{3}$ feet is therefore the ratio of 15 : 22.

Evidently two quantities *different in kind* can have no ratio.

202. Incommensurable Quantities. We cannot expect, however, that two quantities of the same kind chosen at random will have a common measure.

Thus, the side and diagonal of a square have no common measure; for, if the side be a inches long, the diagonal will be $a\sqrt{2}$ inches long, and no measure can be found which will be contained in each an integral number of times.

Again, the diameter and circumference of a circle have no common measure, and are therefore incommensurable.

In this case, as there is no common measure of the two quantities, we cannot find their ratio by the method of § 201. We therefore proceed as follows:

Suppose a and b to be two incommensurable quantities of the *same kind*. Divide b into any integral number, n, equal parts, and suppose one of these parts is contained in a more than m times and less than $m+1$ times. Then $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, and cannot differ from either of these by so much as $\frac{1}{n}$.

But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

Hence, the ratio of two incommensurable quantities cannot be expressed *exactly* by numbers, but it may be expressed *approximately* to any desired degree of accuracy.

Thus, if b represent the side of a square, and a the diagonal,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356.....$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b be taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth*, a *trillionth, or by less than any other assigned value whatever.*

Hence the ratio $\frac{a}{b}$, while it cannot be expressed by numbers *exactly*, may be expressed by numbers as *accurately as we please.*

203. The ratio of two incommensurable quantities is an incommensurable ratio; and is *a fixed value* toward which its successive approximate values constantly tend as the error is made less and less.

204. Equal Incommensurable Ratios. As the treatment of Proportion in Algebra depends upon the assumption that it is possible to find fractions which will represent ratios, and as it appears that no fraction can be found to represent the exact value of an incommensurable ratio, it is necessary to show that *two incommensurable ratios are equal if their approximate values remain equal when the unit of measure is indefinitely diminished.*

Let $a:b$ and $a':b'$ be two incommensurable ratios of which the true values lie between the approximate values $\frac{m}{n}$ and $\frac{m+1}{n}$, when the unit of measure is indefinitely diminished. Then they cannot differ by so much as $\frac{1}{n}$.

Let d denote the difference (if any) between $a:b$ and $a':b'$;

then $$d<\frac{1}{n}.$$

Suppose the fixed value d is not zero; now n can be made as large as we please, and $\frac{1}{n}$ as small as we please; hence $\frac{1}{n}$ can be made less than d if d is not zero.

Therefore $d=0$, and there is no difference between the ratios $a:b$ and $a':b'$. Therefore $a:b=a':b'$.

205. Proportion of Quantities. In order for four quantities, A, B, C, D, to be in proportion, A and B must be of the *same kind*, and C and D of the same kind (but C and D need not necessarily be of the same kind as A and B),

and in addition the ratio of A to B must be the same as the ratio of C to D.

If this be true, we have the proportion

$$A : B = C : D.$$

When four quantities are in proportion, their numerical measures are four abstract numbers in proportion.

206. The laws of § 192, which apply to proportion of abstract numbers, apply to the proportion of concrete quantities, except that alternation will apply only when the four quantities in proportion are *all* of the same kind.

Exercise 34.

1. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of 31 : 17. Find its dimensions.

2. If five gold coins and four silver ones be worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.

3. The lengths of two rectangular fields are in the ratio of 2 : 3, and the breadths in the ratio of 5 : 6. Find the ratio of their areas.

4. Two workmen are paid in proportion to the work they do. A can do in 20 days the work that it takes B 24 days to do. Compare their wages.

5. In a mile race between a bicycle and a tricycle their rates were as 5 : 4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rate of each.

6. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

7. A vessel is half full of a mixture of wine and water. If filled up with wine, the ratio of the quantity of wine to that of water is ten times what it would be if the vessel were filled up with water. Find the ratio of the original quantity of wine to that of water.

8. A quantity of milk is increased by watering in the ratio $4:5$, and then 3 gallons are sold; the remainder is increased in the ratio $6:7$ by mixing it with 3 quarts of water. How many gallons of milk were there at first?

9. Each of two vessels, A and B, contains a mixture of wine and water; A in the ratio of $7:3$, and B in the ratio of $3:1$. How many gallons from B must be put with 5 gallons from A to give a mixture of wine and water in the ratio of $11:4$?

10. The time which an express train takes to travel 180 miles is to that taken by an ordinary train as $9:14$. The ordinary train loses as much time from stopping as it would take to travel 30 miles; the express train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

11. A and B trade with different sums. A gains \$200 and B loses \$50, and now A's stock is to B's as $2:\frac{1}{2}$. But if A had gained \$100 and B lost \$85, their stocks would have been as $15:3\frac{1}{2}$. Find the original stock of each.

12. A line is divided into two parts in the ratio $2:3$, and into two parts in the ratio $3:4$; the distance between the points of section is 2. Find the length of the line.

13. A railway consists of two sections; the annual expenditure on one is increased this year 5%, and on the other 4%, producing on the whole an increase of $4\frac{3}{10}$%. Compare the amounts expended on the two sections last year, and also the amounts expended this year.

VARIATION.

207. A quantity which in any particular problem has a fixed value is called a constant quantity, or simply a **constant**; a quantity which may change its value is called a variable quantity, or simply a **variable.**

Variable numbers, like unknown numbers, are generally represented by x, y, z, etc.; constant numbers, like known numbers, by a, b, c, etc.

208. Two variables may be so related that when a value of one is given, the corresponding value of the other can be found. In this case one variable is said to be a *function* of the other.

Thus, if the rate at which a man walks is known, the distance he walks can be found when the time is given; the distance is in this case a *function* of the time.

209. There is an unlimited number of ways in which two variables may be related. We shall consider in this chapter only a few of these ways.

210. When x and y are so related that their ratio is constant, y is said to vary as x; this is abbreviated thus: $y \propto x$. The sign $\propto$, called the **sign of variation,** is read "varies as."

Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude be changed in any ratio, the area will be changed in the same ratio.

In this case, if we represent the constant ratio by m,

$$y : x = m, \text{ or } \frac{y}{x} = m; \quad \therefore y = mx.$$

Again, if y', x' and y'', x'' be two sets of corresponding values of y and x, then

$$y' : x' = y'' : x'',$$

or

$$y' : y'' = x' : x''.$$

211. When x and y are so related that the ratio of y to $\frac{1}{x}$ is constant, y is said to vary *inversely* as x; this is written $y \propto \frac{1}{x}$.

Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen be doubled, halved, or changed in any ratio, the time required will be halved, doubled, or changed in the inverse ratio.

In this case, $y : \frac{1}{x} = m$; $\therefore y = \frac{m}{x}$, and $xy = m$; that is, the product xy is constant.

As before,

$$y' : \frac{1}{x'} = y'' : \frac{1}{x''},$$

$$x'y' = x''y'',$$

or

$$y' : y'' = x'' : x'.$$

212. If the ratio of $y : xz$ is constant, then y is said to *vary jointly* as x and z.

In this case

$$y = mxz,$$

and

$$y' : y'' = x'z' : x''z''.$$

213. If the ratio $y : \frac{x}{z}$ is constant, then y varies *directly* as x and *inversely* as z.

In this case

$$y = \frac{mx}{z},$$

and

$$y' : y'' = \frac{x'}{z'} : \frac{x''}{z''}.$$

[illegible]

[illegible]

[illegible]

Then [illegible] (1)

Now let [illegible]

Then [illegible] (2)

From (1) and (2),

[illegible] § 196

or [illegible]

or [illegible]

∴ the ratio $\frac{y}{xz}$ is constant, and y varies as xz.

In like manner it may be shown that if y vary as each of any number of quantities x, z, u, etc., when the rest are unchanged, then when they all change, $y \propto xzu$, etc.

Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

The volume of a rectangular solid varies as the length when the width and thickness remain constant; as the width when the length and thickness remain constant; as the thickness when the length and width remain constant; but as the product of length, breadth, and thickness when all three vary.

215. Examples.

(1) If y varies inversely as x, and when $y=2$ the corresponding value of x is 36, find the corresponding value of x when $y=9$.

Here $$y=\frac{m}{x}, \text{ or } m=xy.$$

$$\therefore m=2\times 36=72.$$

If 9 and 72 be substituted for y and m respectively in

$$y=\frac{m}{x},$$

the result is $$9=\frac{72}{x}, \text{ or } 9x=72.$$

$$\therefore x=8. \quad \textit{Ans.}$$

(2) The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weigh 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
D represent the diameter.

Then $W\propto V$ and $V\propto D^3$.

$\therefore W\propto D^3$. § 214, I.

Put $W=mD^3$;

then, since 20 and 4 are corresponding volumes of W and D,

$$20=m\times 64.$$

$$\therefore m=\frac{20}{64}=\frac{5}{16}.$$

$$\therefore W=\tfrac{5}{16}D^3.$$

$\therefore$ when $D=5$, $W=\frac{5}{16}$ of $125=39\frac{1}{16}$.

Exercise 35.

1. If $y \propto x$, and $y = 4$ when $x = 5$, find y when $x = 12$.

2. If $y \propto x$, and when $x = \frac{1}{2}$, $y = \frac{1}{3}$, find y when $x = \frac{1}{3}$.

3. If z vary jointly as x and y, and 3, 4, 5 be simultaneous values of x, y, z, find z when $x = y = 10$.

4. If $y \propto \frac{1}{x}$, and when $y = 10$, $x = 2$, find the value of x when $y = 4$.

5. If $z \propto \frac{x}{y}$, and when $z = 6$, $x = 4$, and $y = 3$, find the value of z when $x = 5$ and $y = 7$.

6. If the square of x vary as the cube of y, and $x = 3$, when $y = 4$, find the equation between x and y.

7. If the square of x vary inversely as the cube of y, and $x = 2$ when $y = 3$, find the equation between x and y.

8. If z vary as x directly and y inversely, and if when $z = 2$, $x = 3$, and $y = 4$, find the value of z when $x = 15$ and $y = 8$.

9. If $y \propto x + c$ where c is constant, and if $y = 2$ when $x = 1$, and if $y = 5$ when $x = 2$, find y when $x = 3$.

10. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it be found that in 3 seconds a stone has fallen 145 feet, and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance fallen at the end of 5 seconds.

11. If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights, and inversely as their sum. If 9 ounces draw 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?

12. The space will also vary as the square of the time. Find the space in Example 11, if the time in the latter case be 3 seconds.

13. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{9}{10}$ ounces.

14. The square of the time of a planet's revolution about the sun varies as the cube of its distance from the sun. The distances of the Earth and Mercury from the sun being 91 and 35 millions of miles, find the time of Mercury's revolution.

15. A spherical iron shell 1 foot in diameter weighs $\frac{91}{216}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.

16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of three spheres whose diameters are 3, 4, 5 inches respectively.

17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

CHAPTER XVI.

PROGRESSIONS.

216. A succession of numbers that proceed according to some fixed law is called a **series**; the successive numbers are called the **terms** of the series.

A series that ends at some particular term is a **finite series**; a series that continues without end is an **infinite series.**

217. The number of different forms of series is unlimited; in this chapter we shall consider only Arithmetical Series, Geometrical Series, and Harmonical Series.

ARITHMETICAL PROGRESSION.

218. A series is called an **arithmetical series** or an **arithmetical progression** when each succeeding term is obtained by adding to the preceding term a *constant difference.*

The general representative of such a series will be

$$a,\ a+d,\ a+2d,\ a+3d\ldots\ldots,$$

in which a is the first term and d the common difference; the series will be *increasing* or *decreasing* according as d is positive or negative.

219. The nth Term. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d will always be one less than the number of the term, so that the *nth* term is $a+(n-1)d.$

If the *nth* term be represented by l, we have

$$l = a + (n-1)\,d. \qquad \text{I.}$$

220. Sum of the Series. If l denote the *nth* term, a the first term, n the number of terms, d the common difference, and s the sum of n terms, it is evident that

$$\begin{aligned} s &= a + (a+d) + (a+2d) + \cdots + (l-d) + l, \text{ or} \\ s &= l + (l-d) + (l-2d) + \cdots + (a+d) + a. \\ \hline \therefore 2s &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \\ &= n\,(a+l). \\ \therefore s &= \frac{n}{2}(a+l). \qquad \text{II.} \end{aligned}$$

221. From the two equations,

$$l = a + (n-1)\,d, \qquad \text{I.}$$

$$s = \frac{n}{2}(a+l), \qquad \text{II.}$$

any *two* of the five numbers a, d, l, n, s may be found when the other *three* are given.

(1) Find the sum of ten terms of the series, 2, 5, 8, 11, ...

Here $a = 2,\ d = 3,\ n = 10.$

From I., $l = 2 + 27 = 29.$

Substituting in II., $s = \frac{10}{2}(2 + 29) = 155.$ *Ans.*

(2) The first term of an arithmetical series is 3, the last term 31, and the sum of the series 136. Find the series.

From I. and II., $31 = 3 + (n-1)\,d,$ (1)

$136 = \frac{n}{2}(3 + 31).$ (2)

From (2), $n = 8.$

Substituting in (1), $d = 4.$

The series is 3, 7, 11, 15, 19, 23, 27, 31.

(3) How many terms of the series, 5, 9, 13, ..., must be taken in order that their sum may be 275?

From I., $$l = 5 + (n-1)\,4;$$
$$\therefore l = 4n + 1. \qquad (1)$$

From II., $$275 = \frac{n}{2}(5 + l). \qquad (2)$$

Substituting in (2) the value of l found in (1),
$$275 = \frac{n}{2}(4n + 6),$$
or $$2n^2 + 3n = 275.$$

We now have to solve this quadratic.

Complete the square,
$$16n^2 + (\ ) + 9 = 2209.$$

Extract the root, $$4n + 3 = \pm 47.$$
$$\therefore n = 11, \text{ or } -12\tfrac{1}{2}.$$

We use only the positive result.

(4) Find n when d, l, s are given.

From I., $$a = l - (n-1)\,d.$$

From II., $$a = \frac{2s - ln}{n}.$$

Therefore, $$l - (n-1)\,d = \frac{2s - ln}{n}.$$
$$\therefore ln - dn^2 + dn = 2s - ln,$$
$$\therefore dn^2 - (2l + d)\,n = -2s.$$

This is a quadratic with n for the unknown number.

Complete the square,
$$4d^2n^2 - (\ ) + (2l + d)^2 = (2l + d)^2 - 8ds.$$

Extract the root,
$$2dn - (2l + d) = \pm\sqrt{(2l + d)^2 - 8ds}.$$
$$\therefore n = \frac{2l + d \pm \sqrt{(2l + d)^2 - 8ds}}{2d}.$$

NOTE. The table on the following page contains the results of the general solution of all possible problems in arithmetical series, in which three of the numbers a, l, d, n, s are given and two required. The student is advised to work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown quantities are represented by letters other than x, y, z.

No.	Given.	Required.	Results.
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{[2ds + (a - \frac{1}{2}d)^2]}.$
3	$a\ n\ s$		$l = \frac{2s}{n} - a.$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$
7	$a\ n\ l$		$s = (l+a)\frac{n}{2}.$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = \frac{2s}{n} - l.$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}.$
14	$a\ n\ s$		$d = \frac{2(s-an)}{n(n-1)}.$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}.$
16	$n\ l\ s$		$d = \frac{2(nl-s)}{n(n-1)}.$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
19	$a\ l\ s$		$n = \frac{2s}{l+a}.$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

222. The **arithmetical mean** between two numbers is the number which stands between them, and makes with them an arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$A - a = b - A.$$

$$\therefore A = \frac{a+b}{2}.$$

223. Sometimes it is required to insert several arithmetical means between two numbers.

Ex. Insert six arithmetical means between 3 and 17.

Here the whole number of terms is eight; 3 is the first term and 17 the eighth.

By I., $$17 = 3 + 7d,$$
$$d = 2.$$

The series is 3, [5, 7, 9, 11, 13, 15,] 17,
the terms in brackets being the means required.

224. When the sum of a number of terms in arithmetical progression is given, it is convenient to represent the terms as follows:

Three terms by $\quad x - y, \quad x, \quad x + y;$

four terms by $\quad x - 3y, \quad x - y, \quad x + y, \quad x + 3y;$

and so on.

Ex. The sum of three numbers in arithmetical progression is 36, and the square of the mean exceeds the product of the two extremes by 49. Find the numbers.

Let $x - y$, x, $x + y$ represent the numbers.

Then, adding, $\quad 3x = 36. \quad \therefore x = 12.$

Putting for x its value, the numbers are

$$12 - y, \quad 12, \quad 12 + y.$$

By the conditions of the problem we have

$$(12)^2 = (12 - y)(12 + y) + 49,$$
$$144 = 144 - y^2 + 49,$$
$$y = \pm 7.$$

The numbers are 5, 12, 19; or 19, 12, 5.

Exercise 36.

Find:

1. The 10th term of 3, 8, 13 .....
2. The 8th term of 12, 9, 6 .....
3. The 12th term of $-4, -9, -14$ .....
4. The 11th term of $2\frac{1}{2}, 1\frac{5}{6}, 1\frac{1}{6}$ .....
5. The 14th term of $1\frac{1}{3}, \dfrac{1}{4}, -\dfrac{5}{6}$ .....

Find the sum of:

6. 8 terms of 4, 7, 10 .....
7. 10 terms of 8, 5, 2 .....
8. 12 terms of -3, 1, 5 .....
9. n terms of $2, 1\frac{1}{6}, \dfrac{1}{3}$ .....
10. n terms of $2\frac{1}{4}, 1\frac{5}{6}, 1\frac{5}{12}$ .....

11. Given $a = 3$, $l = 55$, $n = 13$. Find d and s.
12. Given $a = 3\frac{1}{4}$, $l = 64$, $n = 82$. Find d and s.
13. Given $a = 1$, $n = 20$, $s = 305$. Find d and l.
14. Given $l = 105$, $n = 16$, $s = 840$. Find a and d.
15. Given $d = 7$, $n = 12$, $s = 594$. Find a and l.
16. Given $a = 9$, $d = 4$, $s = 624$. Find n and l.
17. Given $d = 5$, $l = 77$, $s = 623$. Find a and n.

18. When a train arrives at the top of a long slope, the last car is detached and begins to descend, passing over 3 feet in the first second, three times 3 feet in the second second, five times 3 feet in the third second, etc. At the end of 2 minutes it reaches the bottom of the slope. What was its velocity in the last second?

19. Insert eleven arithmetical means between 1 and 12.

20. The first term of an arithmetical series is 3, and the sum of six terms is 28. What term will be 9?

21. How many terms of the series $-5-2+1+\cdots$ must be taken in order that their sum may be 63?

22. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

23. The first term of an arithmetical progression is 3, the third term is 11. Find the sum of seven terms.

24. Arithmetical means are inserted between 8 and 32, so that the sum of the first two is to the sum of the last two as 7 is to 25. How many means are inserted?

25. In an arithmetical series the common difference is 2, and the square roots of the first, third, and sixth terms form a new arithmetical series. Find the series.

26. Find three numbers in arithmetical progression of which the sum is 21, and the sum of the first and second $\frac{3}{4}$ of the sum of the second and third.

27. The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. Find the numbers.

28. The sum of four numbers in arithmetical progression is 12, and the sum of their squares 116. What are these numbers?

29. How many terms of the series 1, 4, 7 ..... must be taken, in order that the sum of the first half may bear to the sum of the second half the ratio 7 : 22?

30. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

31. A man wishes to have his horse shod. The blacksmith asks him $2 a shoe, or 1 cent for the first nail, 3 for the second, 5 for the third, etc. Each shoe has 8 nails. Ought the man to accept the second proposition?

32. A number consists of three digits which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Required the number.

33. There are placed in a straight line upon a lawn 50 eggs 3 feet distant from each other. A person is required to pick them up one by one and carry them to a basket in the line of the eggs and 3 feet from the first egg, while a runner, starting from the basket, touches a goal and returns. At what distance ought the goal to be placed that both men may have the same distance to pass over?

34. Starting from a box, there are placed upon a straight line 40 stones, at the distances 1 foot, 3 feet, 5 feet, etc. A man placed at the box is required to take them and carry them back one by one. What is the total distance that he has to accomplish?

35. The sum of five numbers in arithmetical progression is 45, and the product of the first and fifth is $\frac{5}{8}$ of the product of the second and fourth. Find the numbers.

GEOMETRICAL PROGRESSION.

225. A series is called a **geometrical series** or a **geometrical progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier.*

The general representative of such a series will be

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4 \ldots\ldots,$$

in which a is the first term and r the constant multiplier or ratio.

The terms increase or decrease in numerical magnitude according as r is numerically greater than or numerically less than unity.

226. The nth Term. Since the exponent of r increases by one for each succeeding term after the first, the exponent will always be one less than the number of the term, so that the *nth* term is ar^{n-1}.

If the *nth* term is represented by l, we have

$$l = ar^{n-1}. \qquad \text{I.}$$

227. Sum of the Series. If l represent the *nth* term, a the first term, n the number of terms, r the common ratio, and s the sum of n terms, then

$$s = a + ar + ar^2 + \cdots\cdots \quad ar^{n-1}.$$

Multiply by r,

$$rs = \quad ar + ar^2 + ar^3 + \cdots\cdots ar^{n-1} + ar^n.$$

Subtracting the first equation from the second,

$$rs - s = ar^n - a,$$

or $$(r-1)\,s = a\,(r^n - 1).$$

$$\therefore s = \frac{a\,(r^n - 1)}{r - 1}. \qquad \text{II.}$$

Since $l = ar^{n-1}$, $ar^n = rl$, and II. may be written

$$s = \frac{rl - a}{r - 1}. \qquad \text{III.}$$

228. From the two equations I. and II., or the two equations I. and III., any *two* of the five numbers a, r, l, n, s, may be found when the other *three* are given.

(1) The first term of a geometrical series is 3, the last term 192, and the sum of the series 381. Find the number of terms and the ratio.

From I. and III., $192 = 3r^{n-1}$, (1)

$$381 = \frac{192r - 3}{r - 1}. \qquad (2)$$

From (2), $r = 2$.

Substituting in (1), $2^{n-1} = 64$.

$\therefore n = 7$.

The series is 3, 6, 12, 24, 48, 96, 192.

(2) Find l when r, n, s are given.

From I., $a = \dfrac{l}{r^{n-1}}$.

Substituting in III., $s = \dfrac{rl - \dfrac{l}{r^{n-1}}}{r - 1}$,

$$(r-1)s = \frac{(r^n - 1)}{r^{n-1}} l.$$

$$\therefore l = \frac{(r-1)r^{n-1}s}{r^n - 1}.$$

NOTE. The table on page 186 contains the results of all possible problems in geometrical series in which three of the numbers a, r, l, n, s are given and the other two required, with the exception of those in which n is required; these last require the use of logarithms with which the student is supposed to be not yet acquainted.

No.	Given.	Required.	Results.
1	$a\ r\ n$	l	$l = ar^{n-1}$.
2	$a\ r\ s$		$l = \frac{a + (r-1)s}{r}$.
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
4	$r\ n\ s$		$l = \frac{(r-1)sr^{n-1}}{r^n - 1}$.
5	$a\ r\ n$	s	$s = \frac{a(r^n - 1)}{r-1}$.
6	$a\ r\ l$		$s = \frac{rl - a}{r-1}$.
7	$a\ n\ l$		$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
8	$r\ n\ l$		$s = \frac{lr^n - l}{r^n - r^{n-1}}$.
9	$r\ n\ l$	a	$a = \frac{l}{r^{n-1}}$.
10	$r\ n\ s$		$a = \frac{(r-1)s}{r^n - 1}$.
11	$r\ l\ s$		$a = rl - (r-1)s$.
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$.
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\frac{l}{a}}$.
14	$a\ n\ s$		$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0$.
15	$a\ l\ s$		$r = \frac{s-a}{s-l}$.
16	$n\ l\ s$		$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0$.

229. The **geometrical mean,** between two numbers is the number which stands between them, and makes with them a geometrical series.

If a and b denote two numbers, and G their geometrical mean, then, by the definition of a geometrical series,

$$\frac{G}{a} = \frac{b}{G}.$$

$$\therefore G = \sqrt{ab}.$$

230. Sometimes it is required to insert several geometrical means between two numbers.

Insert three geometrical means between 3 and 48.

Here the whole number of terms is five; 3 is the first term and 48 the fifth.

By I.,
$$48 = 3r^4,$$
$$r^4 = 16,$$
$$r = \pm 2.$$

The series is one of the following.

$$3, \; [\;6, \; 12, \; 24,] \; 48;$$
$$3, \; [-6, \; 12, \; -24,] \; 48.$$

The terms in brackets are the means required.

231. Infinite Geometrical Series. When r is less than 1, the successive terms become numerically smaller and smaller; by taking n large enough we can make the nth term, ar^{n-1}, as small as we please, although we cannot make it absolutely zero.

The sum of n terms, $\frac{rl-a}{r-1}$, may be written $\frac{a}{1-r} - \frac{rl}{1-r}$; this sum differs from $\frac{a}{1-r}$ by the fraction $\frac{rl}{1-r}$; by taking enough terms we can make l, and consequently this difference, as small as we please; the greater the number of terms taken the nearer does their sum approach $\frac{a}{1-r}$. Hence $\frac{a}{1-r}$ is called the *sum* of an infinite number of terms of the series.

$\frac{2}{3}-\frac{2}{3}\left(-\frac{1}{2}\right)^{n}=\frac{2}{3}+\frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}$

(1) Find the sum of the infinite series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\ldots$$

Here, $a=1,\ r=-\frac{1}{2}\cdot$

The sum of the series is $\frac{1}{1+\frac{1}{2}}$ or $\frac{2}{3}\cdot$ *Ans.*

We find for the sum of n terms $\frac{2}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^{n-2}$; this sum evidently approaches $\frac{2}{3}$ as n is increased.

(2) Find the value of the recurring decimal .12135135.....

Consider first the part that recurs; this may be written $\frac{135}{100000}+\frac{135}{100000000}+\ldots\ldots$, and the sum of this series is $\frac{\frac{135}{100000}}{1-\frac{1}{1000}}$, which reduces to $\frac{1}{740}\cdot$ Adding .12, the part that does not recur, we obtain for the value of the decimal $\frac{449}{3700}\cdot$ *Ans.*

Exercise 37.

Find:

1. The eighth term of 3, 6, 12, ...
2. The twelfth term of 2, -4, 8, ...
3. The twentieth term of 1, $-\frac{1}{3}$, $\frac{1}{9}$, ...
4. The eighteenth term of 3, 2, $1\frac{1}{3}$, ...
5. The *nth* term of 1, $-1\frac{1}{3}$, $1\frac{7}{9}$, ...

Find the sum of:

6. Eleven terms of 4, 8, 16, ...
7. Nineteen terms of 9, 3, 1, ...

8. Twelve terms of $5, -3, 1\frac{4}{5}, \ldots\ldots$

9. n terms of $1\frac{7}{8}, \frac{3}{8}, \frac{3}{40}, \ldots\ldots$

Sum to infinity:

10. $4-2+1-\ldots\ldots$

11. $\frac{1}{2}+\frac{1}{3}+\frac{2}{9}+\ldots\ldots$

12. $1-\frac{2}{5}+\frac{4}{25}-\ldots\ldots$

13. $\frac{1}{5}+\frac{1}{15}+\frac{1}{45}+\ldots\ldots$

Find the value of the recurring decimals:

14. $.153153+\ldots\ldots$

15. $.123535+\ldots\ldots$

16. $3.17272+\ldots\ldots$

17. $4.2561561+\ldots\ldots$

18 Given $a=36$, $l=2\frac{1}{4}$, $n=5$. Find r and s.

19. Given $l=128$, $r=2$, $n=7$. Find a and s.

20. Given $r=2$, $n=7$, $s=635$. Find a and l.

21. Given $l=1296$, $r=6$, $s=1555$. Find a and n.

22. Insert three geometrical means between 14 and 224.

23. Insert five geometrical means between 2 and 1458.

24. If the first term is 2 and the ratio 3, what term will be 162?

25. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

26. Four numbers are in geometrical progression; the sum of the first and fourth is 195, and the sum of the second and third is 60. Find the numbers.

27. The sum of four numbers in geometrical progression is 105; the difference between the first and last is to the difference between the second and third in the ratio of 7 : 2. Find the numbers.

28. The first term of an arithmetical progression is 2, and the first, second, and fifth terms are in geometrical progression. Find the sum of 11 terms of the arithmetical progression.

29. The sum of three numbers in arithmetical progression is 6. If 1, 2, 5 be added to the numbers, the three resulting numbers are in geometrical progression. Find the numbers.

30. The sum of three numbers in arithmetical progression is 15; if 1, 4, 19 be added to the numbers, the results are in geometrical progression. Find the numbers.

31. There are four numbers of which the sum is 84; the first three are in geometrical progression and the last three in arithmetical progression; the sum of the second and third is 18. Find the numbers.

32. There are four numbers of which the sum is 13, the fourth being 3 times the second; the first three are in geometrical progression and the last three in arithmetical progression. Find the numbers.

33. The sum of the squares of two numbers exceeds twice their product by 576; the arithmetical mean of the two numbers exceeds the geometrical by 6. Find the numbers.

34. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

35. Find an infinite geometrical series in which each term is 5 times the sum of all the terms that follow it.

36. If a, b, c, d are four numbers in geometrical progression, show that

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

HARMONICAL PROGRESSION.

232. A series is called a **harmonical series,** or a **harmonical progression,** when the reciprocals of its terms form an arithmetical series.

The general representative of such a series will be

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \cdots\cdots,\ \frac{1}{a+(n-1)d}.$$

Questions relating to harmonical series are best solved by writing the reciprocals of its terms, and thus forming an arithmetical series.

233. If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore \frac{2}{H}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\therefore H=\frac{2ab}{a+b}.$$

234. Sometimes it is required to insert several harmonical means between two numbers.

Ex. Insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{19}{72}$, $\frac{14}{72}$, $\frac{9}{72}$; therefore, the harmonical means are $\frac{72}{19}$, $\frac{72}{14}$, $\frac{72}{9}$; or $3\frac{15}{19}$, $5\frac{1}{7}$, 8.

235. Since $A=\frac{a+b}{2}$ and $G=\sqrt{ab}$,

$$H=\frac{G^2}{A} \text{ or } G=\sqrt{AH}.$$

That is, the geometrical mean between two numbers is also the geometrical mean between the arithmetical and harmonical means of the numbers, or

$$A : G = G : H.$$

Hence G lies in numerical value between A and H.

Exercise 38.

1. Insert four harmonical means between 2 and 12.

2. Find two numbers whose difference is 8 and the harmonical mean between them $1\frac{4}{5}$.

3. Find the seventh term of the harmonical series $3, 3\frac{3}{7}, 4 \ldots\ldots$

4. Continue to two terms each way the harmonical series of which two consecutive terms are 15, 16.

5. The first two terms of a harmonical series are 5 and 6. What term will be 30?

6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.

7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{4}{5}$, and one of the numbers is four times the other. Find the numbers.

8. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?

9. The sum of three terms of a harmonical series is 39, and the third is the product of the other two. Find the terms.

10. When a, b, c are in harmonical progression, show that $a : c = a - b : b - c$.

11. If a and b are positive, which is the greater, A or H?

CHAPTER XVII.

SIMPLE INDETERMINATE EQUATIONS.

236. If a single equation involving two unknown numbers be given, and no other condition be imposed, the number of solutions of the equation is unlimited; for if one of the unknown numbers be assumed to have *any* value, a corresponding value of the other may be found.

Such an equation is called an **indeterminate equation.**

Although the number of solutions of an indeterminate equation is unlimited, the values of the unknown numbers are confined to a particular range; this range may be further limited by requiring that the unknown numbers shall be *positive integers.*

237. Every indeterminate equation of the first degree, in which x and y are the unknown numbers, may be made to assume the form

$$ax \pm by = \pm c,$$

where a, b, and c are positive integers and have no common factor.

238. The method of solving an indeterminate equation in positive integers is as follows:

(1) Solve $3x + 4y = 22$, in positive integers.

Transpose, $$3x = 22 - 4y.$$

$$\therefore x = 7 - y + \frac{1-y}{3},$$

the quotient being written as a mixed expression.

$$\therefore x + y - 7 = \frac{1-y}{3}.$$

Since the values of x and y are to be integral, $x+y-7$ will be integral, and hence $\frac{1-y}{3}$ will be integral, though written in the *form* of a fraction.

Let $$\frac{1-y}{3}=m, \text{ an integer.}$$

Then $$1-y=3m.$$
$$\therefore y=1-3m.$$

Substitute this value of y in the original equation,
$$3x+4-12m=22.$$
$$\therefore x=6+4m.$$

The equation $y=1-3m$ shows that m may be 0, or have any negative integral value, but cannot have a positive integral value.

The equation $x=6+4m$ further shows that m may be 0, but cannot have a negative integral value greater than 1.

$$\therefore m \text{ may be } 0 \text{ or } -1,$$

and then $$\left.\begin{matrix} x=6 \\ y=1 \end{matrix}\right\}, \text{ or } \left.\begin{matrix} x=2 \\ y=4 \end{matrix}\right\}.$$

(2) Solve $5x-14y=11$, in positive integers.

Transpose, $$5x=11+14y,$$
$$x=2+2y+\frac{1+4y}{5}. \quad (1)$$
$$\therefore x-2y-2=\frac{1+4y}{5}.$$

Since x and y are to be integral, $x-2y-2$ will be integral, and hence $\frac{1+4y}{5}$ will be integral.

Let $$\frac{1+4y}{5}=m, \text{ an integer.}$$

Then $$y=\frac{5m-1}{4},$$

or $$y=m+\frac{m-1}{4}. \quad (2)$$

Now $\frac{m-1}{4}$ must be integral.

Let $$\frac{m-1}{4}=n, \text{ an integer.}$$

Then $$m=4n+1.$$

Substituting in (2), $y = 5n + 1.$

Substituting in (1), $x = 14n + 5.$

Obviously x and y will both be positive integers if n have *any* positive integral value.

Hence,
$$x = 5,\quad 19,\quad 33,\quad 47,\quad \ldots\ldots$$
$$y = 1,\quad 6,\quad 11,\quad 16,\quad \ldots\ldots$$

Another method of solution is the following:

From the given equation we have $x = \dfrac{11 + 14y}{5}$.

Here y must be so taken that $11 + 14y$ is a multiple of 5; take $y = 1$, then $x = 5$, and we have *one* solution.

Now
$$5x - 14y = 11,$$
and
$$5(5) - 14(1) = 11.$$

Subtract, $5(x-5) - 14(y-1) = 0,$

or
$$\frac{x-5}{y-1} = \frac{14}{5}.$$

Since $x - 5$ and $y - 1$ are integers, $x - 5$ must be the same multiple of 14 that $y - 1$ is of 5.

Hence, if $x - 5 = 14m$, then $y - 1 = 5m$.

$$\therefore x = 14m + 5, \text{ and } y = 5m + 1.$$

Therefore
$$x = 5,\quad 19,\quad 33,\quad 47,\quad \ldots\ldots$$
$$y = 1,\quad 6,\quad 11,\quad 16,\quad \ldots\ldots$$

It will be seen from (1) and (2) that when only positive integers are required, the number of solutions will be *limited* or *unlimited* according as the sign connecting x and y is *positive* or *negative.*

(3) Find the least number that when divided by 14 and 5 will give remainders 1 and 3 respectively.

If N represent the number, then

$$\frac{N-1}{14} = x, \text{ and } \frac{N-3}{5} = y.$$
$$\therefore N = 14x + 1, \text{ and } N = 5y + 3.$$
$$\therefore 14x + 1 = 5y + 3.$$
$$5y = 14x - 2,$$
$$5y = 15x - 2 - x.$$
$$\therefore y = 3x - \frac{2+x}{5}.$$

Let
$$\frac{2+x}{5} = m, \text{ an integer.}$$
$$\therefore x = 5m - 2.$$

3

$$y = \tfrac{1}{5}(14x - 2), \text{ from original equation.}$$
$$\therefore y = 14m - 6.$$
If $m = 1$, $\quad x = 3$, and $y = 8$,
$$\therefore N = 14x + 1 = 5y + 3 = 43. \quad \textit{Ans.}$$

(4) Solve $5x + 6y = 30$, so that x may be a multiple of y, and both x and y positive.

Let $\quad x = my.$

Then $\quad (5m + 6)y = 30.$

$$\therefore y = \frac{30}{5m + 6},$$

and
$$x = \frac{30m}{5m + 6}.$$

If $m = 2$, $\quad x = 3\tfrac{3}{4}, \ y = 1\tfrac{7}{8}.$

If $m = 3$, $\quad x = 4\tfrac{2}{7}, \ y = 1\tfrac{3}{7}.$

(5) Solve $14x + 22y = 71$, in positive integers.

$$x = 5 - y + \frac{1 - 8y}{14}.$$

If we multiply the fraction by 7 and reduce, the result is

$$-4y + \tfrac{1}{2},$$

a form which shows that there can be no *integral* solution.

There can be no integral solution of $ax \pm by = \pm c$ if a and b have a common factor not common also to c; for, if d be a factor of a and also of b, but not of c, the equation may be written

$$mdx \pm ndy = \pm c, \text{ or } nx \pm ny = \pm \frac{c}{d};$$

which is impossible, since $\frac{c}{d}$ is a fraction, and $mx \pm ny$ is an integer, if x and y are integers.

Exercise 39.

Solve in positive integers:

1. $x + y = 12.$
2. $2x + 11y = 83.$
3. $4x + 9y = 53.$
4. $8x + 5y = 74.$
5. $5x + 3y = 105.$
6. $\tfrac{3}{4}x + 5y = 92.$

7. $\frac{3}{4}x + \frac{1}{4}y = 27.$

8. $\frac{2}{5}x + \frac{3}{4}y = 53.$

Solve in least possible integers:

9. $7x - 2y = 12.$

10. $9x - 5y = 21.$

11. $7x - 4y = 45.$

12. $11x - 5y = 73.$

13. $15x - 47y = 11.$

14. $23x - 14y = 99.$

15. Find two numbers which, multiplied respectively by 7 and 17, have for the sum of their products 1135.

16. If two numbers are multiplied respectively by 8 and 17, the difference of their products is 10. What are the numbers?

17. If two numbers are multiplied respectively by 7 and 15, the first product is greater by 12 than the second. Find the numbers.

18. Divide 89 in two parts, one of which is divisible by 3, and the other by 8.

19. Divide 314 in two parts, one of which is a multiple of 11, and the other a multiple of 13.

20. What is the smallest number which, divided by 5 and by 7, gives each time 4 for a remainder?

21. The difference of two numbers is 151. The first divided by 8 has 5 for a remainder, and 4 must be added to the second to make it divisible by 11. What are the numbers?

22. Find pairs of fractions whose denominators are 24 and 16, and whose sum is $\frac{19}{24}$.

23. How can one pay a sum of \$87, giving only bills of \$5 and \$2?

24. A man buys calves at \$5 apiece, and pigs for \$3 apiece. He spends in all \$114. How many did he buy of each?

25. A person bought 40 animals, consisting of pigs, geese, and chickens, for \$40. The pigs cost \$5 apiece, the geese \$1, and the chickens 25 cents each. Find the number he bought of each.

26. Solve $18x - 5y = 70$ so that y may be a multiple of x, and both positive.

27. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.

28. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8, respectively, and the sum of the quotients shall be 10.

29. In how many ways can \$3.60 be paid with dollars and twenty-cent pieces?

30. In how many ways can 300 pounds be weighed with 7 and 9 pound weights?

31. Find the general form of the numbers that, divided by 2, 3, 7, have for remainders 1, 2, 5, respectively.

32. Find the general form of the numbers that, divided by 7, 8, 9, have for remainders 6, 7, 8, respectively.

33. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the total cost £100. If the oxen cost £5, the sheep £1, and the hens 1*s*. each, how many of each did he buy?

34. A farmer sells 15 calves, 14 lambs, and 13 pigs, and receives \$200. Some days after, at the same price, he sells 7 calves, 11 lambs, and 16 pigs, for which he receives \$141. What is the price of each?

CHAPTER XVIII.

BINOMIAL THEOREM.

239. Binomial Theorem, Positive Integral Exponent. By successive multiplication we obtain the following identities:

$$(a+b)^2 \equiv a^2 + 2ab + b^2;$$

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3;$$

$$(a+b)^4 \equiv a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

The expressions on the right may be written in a form better adapted to show the law of their formation:

$$(a+b)^2 \equiv a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2;$$

$$(a+b)^3 \equiv a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3;$$

$$(a+b)^4 \equiv a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4.$$

NOTE. The dot between the Arabic figures means the same as the sign $\times$.

240. Let n represent the exponent of $(a+b)$ in any one of these identities; then, in the expressions on the right, we observe that the following laws hold true:

I. The number of terms is $n+1$.

II. The first term is a^n, and the exponent of a is one less in each succeeding term.

The first power of b occurs in the second term, the second power in the third term, and the exponent of b is one greater in each succeeding term.

The sum of the exponents of a and b in any term is n.

III. The coefficient of the first term is 1; of the second term, n; of the third term, $\frac{n(n-1)}{1\cdot 2}$; of the fourth term, $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$; and so on.

241. Consider the coefficient of any term: the number of factors in the numerator is the same as the number of factors in the denominator, and the number of factors in each is the same as the exponent of b in that term; this exponent is one less than the number of the term.

242. Proof of the Theorem. That the laws of § 240 hold true when the exponent is *any* positive integer, is shown as follows:

We know that the laws hold for the fourth power; suppose, for the moment, that they hold for the kth power.

We shall then have

$$(a+b)^k \equiv a^k + ka^{k-1}b + \frac{k(k-1)}{1\cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3 + \cdots\cdots \quad (1)$$

Multiply both members of (1) by $a+b$; the result is

$$(a+b)^{k+1} \equiv a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots\cdots \quad (2)$$

In the right member of (1) for k put $k+1$; this gives

$$a^{k+1} + (k+1)a^k b + \frac{(k+1)(k+1-1)}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)(k+1-1)(k+1-2)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots\cdots$$

This last expression, simplified, is seen to be identical with the right member of (2), and this in turn by (2) is identical with $(a+b)^{k+1}$.

Hence (1) holds when for k we put $k+1$; that is, if the laws of § 240 hold for the kth power, they must hold for the $(k+1)th$ power.

But the laws hold for the fourth power; therefore they must hold for the fifth power.

Holding for the fifth power, they must hold for the sixth power; and so on for any positive integral power.

Therefore they must hold for the nth power, if n is a positive integer; and we have

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots\ldots \qquad \mathbf{A}$$

NOTE. The proof of § 242 is an example of a proof by *mathematical induction.*

243. This formula is known as the **binomial theorem.**

The expression on the right is known as the **expansion** of $(a+b)^n$; this expansion is a *finite series* when n is a positive integer. That the series is finite may be seen as follows:

In writing out the successive coefficients we shall finally arrive at a coefficient which contains the factor $n-n$; the corresponding term will vanish. The coefficients of the succeeding terms likewise all contain the factor $n-n$, and all these terms will vanish.

244. If a and b be interchanged, the identity **A** may be written

$$(a+b)^n \equiv (b+a)^n \equiv b^n + nb^{n-1}a + \frac{n(n-1)}{1\cdot 2}b^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}b^{n-3}a^3 + \ldots\ldots$$

This last expansion is the expansion of **A** written in reverse order. Comparing the two expansions we see that: the coefficient of the last term is the same as the coefficient of the first term; the coefficient of the last term but one is the same as the coefficient of the first term but one; and so on.

In general, the coefficient of the *rth* term from the end is the same as the coefficient of the *rth* term from the beginning. In writing out an expansion by the binomial theorem, after arriving at the middle term, we can shorten the work by observing that the remaining coefficients are those already found written in reverse order.

245. If b be negative, the terms which involve even powers of b will be positive, and those which involve odd powers of b negative. Hence,

$$(a-b)^n \equiv a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots \quad \mathbf{B}$$

Also, putting 1 for a and x for b, in **A** and **B**,

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \quad \mathbf{C}$$

$$(1-x)^n \equiv 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots \quad \mathbf{D}$$

246. Examples:

(1) Expand $(1+2x)^5$.

In **C** for x put $2x$, and for n put 5. The result is

$$(1+2x)^5 \equiv 1 + 5(2x) + \frac{5\cdot 4}{1\cdot 2}4x^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}8x^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}16x^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}32x^5$$

$$\equiv 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

(2) Expand to three terms $\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6$.

Put a for $\frac{1}{x}$, and b for $\frac{2x^2}{3}$; then, by **B**,

$$(a-b)^6 \equiv a^6 - 6a^5b + 15a^4b^2 + \ldots\ldots$$

Replacing a and b by their values,

$$\left(\frac{1}{x} - \frac{2x^2}{3}\right)^6 \equiv \left(\frac{1}{x}\right)^6 - 6\left(\frac{1}{x}\right)^5\left(\frac{2x^2}{3}\right) + 15\left(\frac{1}{x}\right)^4\left(\frac{2x^2}{3}\right)^2 - \ldots\ldots$$

$$\equiv \frac{1}{x^6} - \frac{4}{x^3} + \frac{20}{3} - \ldots\ldots$$

247. Any Required Term. From **A** it is evident (§ 241) that the $(r+1)th$ term in the expansion of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots \text{ to } r \text{ factors}}{1\cdot 2\cdot 3\ldots\ldots r}a^{n-r}b^r.$$

The $(r+1)th$ term in the expansion of $(a-b)^n$ is the same as the above if r be even, and the negative of the above if r be odd.

Ex. Find the eighth term of $\left(4 - \frac{x^2}{2}\right)^{10}$.

Here $\qquad a = 4,\ b = \frac{x^2}{2},\ n = 10,\ r = 7.$

The term required is

$$-\frac{10\cdot 9\cdot 8\ 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(4)^3\left(\frac{x^2}{2}\right)^7,$$

which reduces to $\quad -60x^{14}$.

248. The Greatest Coefficient. Suppose that the coefficient of the $(r+1)th$ term is the numerically greatest coefficient.

This coefficient, the preceding and following coefficients, are the following:

$r\,th$ term, $$\frac{n(n-1)\cdots\cdots(n-r+2)}{1\cdot 2\cdot 3\ldots\ldots(r-1)};$$

$(r+1)th$ term, $$\frac{n(n-1)\cdots\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\ldots\ldots(r-1)r};$$

$(r+2)th$ term, $$\frac{n(n-1)\cdots\cdots(n-r+2)(n-r+1)(n-r)}{1\cdot 2\cdot 3\ldots\ldots(r-1)r(r+1)}.$$

The first coefficient may be obtained by multiplying the second by $\frac{r}{n-r+1}$; the third, by multiplying the second by $\frac{n-r}{r+1}$. If the second coefficient is numerically the greatest,

$$\frac{r}{n-r+1}<1, \qquad \text{and } \frac{n-r}{r+1}<1;$$

$$r<n-r+1, \text{ and } r+1>n-r;$$

$$r<\frac{n+1}{2}, \qquad \text{and} \qquad r>\frac{n-1}{2}.$$

If n is even, $r=\frac{n}{2}$, and $r+1=\frac{n+2}{2}$; in this case there is one middle term and its coefficient is the greatest coefficient.

If n is odd, we can only have $r=\frac{n+1}{2}$, or $r=\frac{n-1}{2}$; in this case there are two middle terms; their coefficients are alike, and are the two greatest coefficients.

249. A **trinomial** may be expanded by the binomial theorem as follows:

Expand $(1+2x-x^2)^3$.

Put $\qquad 2x-x^2=z;$

then $\qquad (1+z)^3=1+3z+3z^2+z^3.$

$$\therefore (1+2x-x^2)^3=1+3(2x-x^2)+3(2x-x^2)^2+(2x-x^2)^3$$
$$=1+6x+9x^2-4x^3-9x^4+6x^5-x^6.$$

Exercise 40.

Expand:

1. $(1+3x)^5$.
4. $(2+x^2)^6$.
7. $(3x-2y)^6$.
2. $\left(1+\frac{2x}{3}\right)^4$.
5. $\left(\frac{2}{x}-\frac{x^2}{4}\right)^5$.
8. $\left(\frac{2x^2}{y}-\frac{\sqrt[3]{y^2}}{4}\right)^5$.
3. $\left(1-\frac{\sqrt{x^3}}{3}\right)^4$.
6. $\left(\frac{2a}{x}-\frac{x^2}{(2a)^2}\right)^5$
9. $\left(\sqrt{\frac{a^3}{b^2}}-\frac{\sqrt[4]{b^3}}{4a}\right)^4$.
10. $(1+4x+3x^2)^4$.
11. $(a^2-ax-2x^2)^3$.

Find:

12. The fourth term of $\left(x+\frac{1}{2x}\right)^8$.
13. The eighth term of $\left(2-\frac{1}{4x^2}\right)^{10}$.
14. The twelfth term of $\left(\frac{1}{x}-\frac{\sqrt{x}}{4}\right)^{14}$.
15. The twentieth term of $\left(x-\frac{2}{3\sqrt[4]{x}}\right)^{23}$.
16. The fourteenth term of $\left(\sqrt[3]{x^2}-\frac{1}{2\sqrt{x}}\right)^{17}$.
17. The $(r+1)th$ term of $\left(\sqrt{x}+\sqrt[3]{\frac{3}{2x}}\right)^8$.
18. The $(r+1)th$ term of $\left(\sqrt{\frac{1}{3x}}-\frac{\sqrt{x}}{2}\right)^{10}$.
19. The $(r+3)th$ term of $\left(\frac{x}{2y}-\frac{y}{\sqrt{3x}}\right)^{12}$.
20. Find the middle term of $\left(\frac{3}{4x}-\sqrt{\frac{x^3}{2}}\right)^{12}$.

21. Find the two middle terms of $\left(\frac{a}{\sqrt{2x}}+\sqrt{\frac{3x}{4a}}\right)^{15}$.

22. Find the rth term from the end of $\left(\frac{\sqrt[3]{x^2}}{4}-\sqrt{\frac{x^3}{2}}\right)^{11}$.

23. In the expansion of $(a+b)^n$ show that the sum of the coefficients is 2^n.

24. In the expansion of $(a+b)^n$ show that the sum of the even coefficients is equal to the sum of the odd coefficients.

25. Expand

$$\left(x+\frac{\sqrt{-1}}{2x}\right)^4;\ \left(\sqrt{-1}+\frac{\sqrt[3]{x}}{4\sqrt{-1}}\right)^6;\ \left(\frac{\sqrt{-a}}{2}+\frac{1}{a\sqrt{-1}}\right)^5.$$

26. If A is the sum of the odd terms, and B the sum of the even terms, in the expansion of $(a+b)^n$, show that

$$A^2-B^2=(a^2-b^2)^n.$$

250. Convergent and Divergent Series. By performing the indicated division, we obtain from the fraction $\frac{1}{1-x}$ the infinite series $1+x+x^2+x^3+\cdots\cdots$ This series, however, is not equal to the fraction for all values of x.

251. Suppose x numerically less than 1. In this case we can obtain an approximate value for the sum of the series by taking the sum of a number of terms; the greater the number of terms taken, the nearer will this approximate sum approach the value of the fraction. The approximate sum will never be exactly equal to the fraction, however great the number of terms taken; but by taking enough terms, it can be made to differ from the fraction as little as we please.

Thus, if $x = \frac{1}{2}$, the fraction is $\dfrac{1}{1-\frac{1}{2}} = 2$, and the series is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$$

The sum of four terms of this series is $1\frac{7}{8}$; the sum of five terms, $1\frac{15}{16}$; the sum of six terms, $1\frac{31}{32}$; and so on. The successive approximate sums approach, but never reach, the finite value 2.

When x is numerically less than 1, the series is equal to the fraction, and we have the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$$

252. A series is said to be **convergent** when the sum of a number of terms, as the number of terms is indefinitely increased, approaches some fixed finite value; this finite value is called the **sum** of the series.

~~Any finite series is convergent, since its sum is simply the sum of all its terms.~~

253. In the series $1 + x + x^2 + x^3 + \cdots\cdots$ suppose x numerically greater than 1. In this case, the greater the number of terms taken, the greater will their sum be; by taking enough terms we can make their sum as large as we please. The fraction, on the other hand, has a definite value. Hence, when x is numerically greater than 1, the series is *not* equal to the fraction, and we *do not* have the equation $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$

Thus, if $x = 2$, the fraction is $\dfrac{1}{1-2} = -1$; the series is

$$1 + 2 + 4 + 8 + \cdots\cdots$$

The greater the number of terms taken, the larger will their sum be. Evidently the fraction and the series are not equal.

254. In the same series suppose $x = 1$. In this case the fraction is $\frac{1}{1-1} = \frac{1}{0}$, and the series $1 + 1 + 1 + 1 + \cdots\cdots$ The more terms we take, the greater will the sum of the series be. We do not know whether or not the fraction is equal to the series.

If x, however, is not exactly 1, but is a little less than 1, the value of the fraction $\frac{1}{1-x}$ will be very great, and the sum of the series also very great; and the fraction will be equal to the series.

Suppose $x = -1$. In this case the fraction is $\frac{1}{1+1} = \frac{1}{2}$, and the series $1 - 1 + 1 - 1 + \cdots\cdots$ If we take an even number of terms, their sum is 0; if an odd number, their sum is 1.

Hence, when $x = -1$, the fraction is *not* equal to the series.

255. A series is said to be **divergent** when the sum of a number of terms, as the number of terms is indefinitely increased, either increases without end, or oscillates in value without approaching any finite value.

No reasoning can be based on a divergent series; hence, in using an infinite series it is necessary to make such restrictions as will cause the series to be convergent.

Thus we can use the infinite series $1 + x + x^2 + x^3 + \cdots\cdots$ when, and only when, x lies between $+1$ and -1.

Observe that any series of the form $A + Bx + Cx^2 + \cdots\cdots$ is convergent when $x = 0$, since in this case the series reduces to the first term.

256. Identical Series. *If two series which are arranged by powers of x be equal for all values of x which make both series convergent, the corresponding coefficients are equal each to each.*

Let the equation

$$a + bx + cx^2 + dx^3 + \cdots\cdots = A + Bx + Cx^2 + Dx^3 + \cdots\cdots \quad (1)$$

hold true for all values of x which make both series convergent.

Since this equation holds true for all values of x which make both series convergent, it will hold true when $x = 0$.

Let $x = 0$; then $\qquad a = A. \qquad (2)$

Subtract (2) from (1) and divide both members by x; then

$$b + cx + dx^2 + \cdots\cdots = B + Cx + Dx^2 + \cdots\cdots \quad (3)$$

Let $x = 0$; then $\qquad b = B. \qquad (4)$

Subtract (4) from (3) and divide both members by x; then

$$c + dx + \cdots\cdots = C + Dx + \cdots\cdots$$

Let $x = 0$; then $\qquad c = C.$

And so on.

257. Binomial Theorem, Any Exponent. We have seen (§ 245) that when n is a positive integer we have the identity

$$(1 + x)^n \equiv 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots\cdots$$

We proceed to the case of fractional and negative exponents.

I. Suppose n is a positive fraction, $\frac{p}{q}$. We may assume that

$$(1 + x)^p = (A + Bx + Cx^2 + Dx^3 + \cdots\cdots)^q, \quad (1)$$

provided x be so taken that the series

$$A + Bx + Cx^2 + Dx^3 + \cdots\cdots$$

is convergent (§ 252).

That this assumption is allowable may be seen as follows:

Expand both members of (1).

We obtain

$$1 + px + \frac{p(p-1)}{1\cdot 2}x^2 + \frac{p(p-1)(p-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots,$$

and

$$A^q + qA^{q-1}Bx + \left.\begin{array}{l}\frac{q(q-1)}{1\cdot 2}A^{q-2}B^2 \\ + qA^{q-1}C\end{array}\right| x^2 + \left.\begin{array}{l}\frac{q(q-1)(q-2)}{1\cdot 2\cdot 3}A^{q-3}B^3 \\ + q(q-1)A^{q-2}BC + qA^{q-1}D\end{array}\right| x^3 +$$

In the first k coefficients of the second series there enter only the first k of the coefficients A, E, C, D, ... If, then, we equate the coefficients of corresponding terms in the two series (§ 256) as far as the kth term, we shall have just k equations to find k unknown numbers A, B, C, D, ... Hence the assumption made in (1) is allowable.

Comparing the two first terms and the two second terms, we obtain

$$A^q = 1, \qquad \therefore A = 1;$$

$$qA^{q-1}B = p, \qquad \therefore B = \frac{p}{q}.$$

Extracting the qth root of both members of (1), we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + Cx^2 + Dx^3 + \cdots\cdots \tag{2}$$

where x is to be so taken that the series on the right is convergent.

II. Suppose n is a negative number, integral or fractional. Let $n = -m$, so that m is positive; then

$$(1+x)^n = (1+x)^{-m} = \frac{1}{(1+x)^m}.$$

From (2), whether m is integral or fractional, we may assume

$$\frac{1}{(1+x)^m} = \frac{1}{1 + mx + cx^2 + dx^3 + \cdots\cdots}.$$

By actual division this gives an equation in the form

$$(1+x)^{-m} = 1 - mx + Cx^2 + Dx^3 + \cdots\cdots \tag{3}$$

258. It appears from (2) and (3) that whether n be integral or fractional, positive or negative, we may assume

$$(1+x)^n = 1 + nx + Cx^2 + Dx^3 + \cdots\cdots,$$

provided the series on the right is convergent.

Squaring both members,

$$(1+2x+x^2)^n = 1 + 2nx + \left.\begin{matrix}2C \\ +n^2\end{matrix}\right| x^2 + \left.\begin{matrix}2D \\ +2nC\end{matrix}\right| x^3 + \cdots\cdots \quad (1)$$

Also, since

$$(1+y)^n = 1 + ny + Cy^2 + Dy^3 + \cdots\cdots,$$

we have, putting $2x + x^2$ for y,

$$\begin{aligned}(1+2x+x^2)^n &= 1 + n(2x+x^2) + C(2x+x^2)^2 \\ &\qquad + D(2x+x^2)^3 \cdots\cdots \\ &= 1 + 2nx + \left.\begin{matrix}n \\ +4C\end{matrix}\right| x^2 + \left.\begin{matrix}4C \\ +8D\end{matrix}\right| x^3 + \cdots\cdots \quad (2)\end{aligned}$$

Comparing corresponding coefficients in (1) and (2),

$$n + 4C \quad = 2C + n^2,$$

$$4C + 8D = 2D + 2nC.$$

$$\therefore 2C = n^2 - n, \qquad C = \frac{n(n-1)}{1 \cdot 2},$$

$$3D = (n-2)C, \quad D = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

and so on.

Hence, whether n be integral or fractional, positive or negative, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots\cdots$$

provided, always, x be so taken that the series on the right is convergent.

The series obtained will be an infinite series unless n is a positive integer (§ 243).

259. If x is negative,

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

Also, if $x < a$,

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$$
$$= a^n\left(1 + n\frac{x}{a} + \frac{n(n-1)}{1\cdot 2}\frac{x^2}{a^2} + \cdots\cdots\right)$$
$$= a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \cdots\cdots;$$

if $x > a$,

$$(a+x)^n = (x+a)^n = x^n\left(1+\frac{a}{x}\right)^n$$
$$= x^n\left(1 + n\frac{a}{x} + \frac{n(n-1)}{1\cdot 2}\frac{a^2}{x^2} + \cdots\cdots\right)$$
$$= x^n + nax^{n-1} + \frac{n(n-1)}{1\cdot 2}a^2x^{n-2} + \cdots\cdots$$

260. Examples.

(1) Expand $(1+x)^{\frac{1}{3}}$.

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\cdot 2}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$
$$= 1 + \tfrac{1}{3}x - \frac{2}{3\cdot 6}x^2 + \frac{2\cdot 5}{3\cdot 6\cdot 9}x^3 - \cdots\cdots$$

The above equation is only true for those values of x which make the series convergent.

(2) Expand $\dfrac{1}{\sqrt[4]{1-x}}$.

$$\frac{1}{\sqrt[4]{1-x}} = (1-x)^{-\frac{1}{4}}$$
$$= 1 - (-\tfrac{1}{4})x + \frac{-\frac{1}{4}\cdot -\frac{5}{4}}{1\cdot 2}x^2 - \frac{-\frac{1}{4}\cdot -\frac{5}{4}\cdot -\frac{9}{4}}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$
$$= 1 + \tfrac{1}{4}x + \frac{1\cdot 5}{4\cdot 8}x^2 + \frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}x^3 + \cdots\cdots$$

if x is so taken that the series is convergent.

A root may often be extracted by means of an expansion.

(3) Extract the cube root of 344 to six decimal places.

$$344 = 343\left(1+\frac{1}{343}\right) = 7^3\left(1+\frac{1}{343}\right).$$

$$\begin{aligned} \therefore \sqrt[3]{344} &= 7\left(1+\frac{1}{343}\right)^{\frac{1}{3}}, \\ &= 7\left(1+\frac{1}{3}\left(\frac{1}{343}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\cdot 2}\left(\frac{1}{343}\right)^2+\cdots\cdots\right), \\ &= 7(1+.000971815-.000000944), \\ &= 7.006796. \end{aligned}$$

(4) Find the eighth term of $\left(x-\dfrac{3}{4\sqrt{x}}\right)^{-\frac{1}{2}}$.

Here (§ 147)

$$a = x, \quad b = \frac{3}{4\sqrt{x}} = \frac{3}{4x^{\frac{1}{2}}}, \quad n = -\frac{1}{2}, \quad r = 7.$$

The term is

$$-\frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}\cdot-\frac{7}{2}\cdot-\frac{9}{2}\cdot-\frac{11}{2}\cdot-\frac{13}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^{-\frac{15}{2}}\left(\frac{3}{4x^{\frac{1}{2}}}\right)^7,$$

or

$$\frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 3^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 4^7\cdot x^{11}}.$$

Exercise 41.

Expand to four terms:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{\frac{1}{4}}$.
3. $\dfrac{1}{\sqrt[3]{1-x}}$.
4. $(1-x)^{-4}$.
5. $(1+x)^{\frac{5}{2}}$.
6. $\dfrac{1}{\sqrt[4]{a^2-x^2}}$.
7. $\sqrt[5]{2-3x}$.
8. $\sqrt[3]{(2-x^2)^2}$.
9. $\dfrac{1}{\sqrt[4]{(1+2x^2)^3}}$.

Find:

10. The eighth term of $(1-2x)^{\frac{1}{2}}$.
11. The tenth term of $(a-3x)^{-\frac{2}{3}}$.
12. The $(r+1)$th term of $(a+x)^{\frac{1}{3}}$.

13. The $(r+1)$th term of $(a^2-4x^2)^{-\frac{5}{2}}$.

14. Find $\sqrt{65}$ to five decimal places.

15. Find $\sqrt[7]{129}$ to six decimal places.

16. Expand $(1-2x+3x^2)^{-\frac{2}{3}}$ to four terms.

17. Find the coefficient of x^4 in the expansion of $\dfrac{(1+2x)^2}{(1+3x)^3}$.

18. By means of the expansion of $(1+x)^{\frac{1}{2}}$ show that the limit of the series

$$1+\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1\cdot 3}{2\cdot 3\cdot 2^3}-\frac{1\cdot 3\cdot 5}{2\cdot 3\cdot 4\cdot 2^4}+\cdots\cdots$$

is $\sqrt{2}$.

19. Find the first negative term in the expansion of $(1+x)^{\frac{11}{3}}$.

20. Expand $\sqrt{\dfrac{1+x}{1-x}}$ in ascending powers of x to six terms.

21. If n is a positive integer, show that the coefficient of x^{n-1}, in the expansion of $(1-x)^{-n}$, is always twice the coefficient of x^{n-2}.

22. If m and n are positive integers, show that the coefficient of x^m in $(1-x)^{-n-1}$ is the same as the coefficient of x^n in $(1-x)^{-m-1}$.

23. Find the coefficient of x^{3r} in the expansion of $\sqrt{\dfrac{1-x}{1+x}}$ in ascending powers of x.

24. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{1\cdot 2\cdot 3\cdots\cdots 2r}{(1\cdot 2\cdot 3\cdots\cdots r)^2}$.

CHAPTER XIX.

LOGARITHMS.

261. Definitions. Let any positive number be selected as a **base**; let all other numbers be regarded as powers of this base. Then, the exponent of the power to which the base must be raised to obtain a given number is called the **logarithm** of that number to the given base.

Any positive number may be selected as the base; and to each base corresponds a **system of logarithms.**

Thus, since $2^3 = 8$, the logarithm of 8 in the system of which 2 is the base is 3.

That is, the logarithm of 8 to the base 2 is 3; this is abbreviated $\log_2 8 = 3$.

In general, if $a^n = N$, then $n = \log_a N$.

Observe that $a^n = N$ and $n = \log_a N$ are two different ways of expressing the same relation between n and N. The identity, $a^{\log_a N} \equiv N$, is sometimes useful.

The subscript which gives the base is omitted when there is no uncertainty as to what number is being used as the base.

In this chapter by $\sqrt[n]{a}$ will be meant the positive real value of the root; consequently, in a system with a positive base, negative numbers cannot have real logarithms.

262. The logarithms of such numbers as are perfect powers of the base selected are commensurable numbers; the logarithms of all other numbers are incommensurable numbers.

Remark. By an incommensurable number is meant a number which has no common measure with unity (§ 202).

Incommensurable logarithms are expressed approximately to any desired degree of accuracy by means of decimal fractions.

263. A logarithm will generally consist of two parts, an integral part and a fractional part; the integral part is called the **characteristic**, and the fractional part the **mantissa.**

The calculation of logarithms to a given base will be considered in a later chapter.

264. Incommensurable Exponents. It will now be necessary to prove that the laws which in Chapter VIII. were found to apply to commensurable exponents apply also to incommensurable exponents.

Let a be *any positive number*, and let m and n be two positive incommensurable numbers.

To prove $$a^m a^n = a^{m+n}.$$

We can always find (§ 202) four positive integers, p, q, r, s, such that m lies between $\frac{p}{q}$ and $\frac{p+1}{q}$, and n between $\frac{r}{s}$ and $\frac{r+1}{s}$.

Then a^m lies between $a^{\frac{p}{q}}$ and $a^{\frac{p+1}{q}}$, and a^n lies between $a^{\frac{r}{s}}$ and $a^{\frac{r+1}{s}}$.

Therefore $a^m a^n$ lies between $a^{\frac{p}{q}}a^{\frac{r}{s}}$ and $a^{\frac{p+1}{q}}a^{\frac{r+1}{s}}$.

But, $$a^{\frac{p}{q}}a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}},$$

and $$a^{\frac{p+1}{q}}a^{\frac{r+1}{s}} = a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}}.$$

Hence, $a^m a^n$ lies between $a^{\frac{p}{q}+\frac{r}{s}}$ and $a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}}$, and consequently differs from $a^{\frac{p}{q}+\frac{r}{s}}$ by less than $(a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}} - a^{\frac{p}{q}+\frac{r}{s}})$; that is, by less than $a^{\frac{p}{q}+\frac{r}{s}}(a^{\frac{1}{q}+\frac{1}{s}} - 1)$.

Also, since m lies between $\frac{p}{q}$ and $\frac{p+1}{q}$, and n between $\frac{r}{s}$ and $\frac{r+1}{s}$, a^{m+n} lies between $a^{\frac{p}{q}+\frac{r}{s}}$ and $a^{\frac{p}{q}+\frac{r}{s}+\frac{1}{q}+\frac{1}{s}}$; and consequently differs from $a^{\frac{p}{q}+\frac{r}{s}}$ by less than $a^{\frac{p}{q}+\frac{r}{s}}(a^{\frac{1}{q}+\frac{1}{s}}-1)$.

Therefore, the expressions $a^m a^n$ and a^{m+n} have the same approximate value $a^{\frac{p}{q}+\frac{r}{s}}$, and each differs from this value by less than $a^{\frac{p}{q}+\frac{r}{s}}(a^{\frac{1}{q}+\frac{1}{s}}-1)$.

Now let q and s be continually increased, p and r being always so taken that m lies between $\frac{p}{q}$ and $\frac{p+1}{q}$, and n between $\frac{r}{s}$ and $\frac{r+1}{s}$. Then, $\frac{1}{q}$ and $\frac{1}{s}$ continually decrease; $a^{\frac{1}{q}+\frac{1}{s}}$ approaches 1; and $a^{\frac{p}{q}+\frac{r}{s}}(a^{\frac{1}{q}+\frac{1}{s}}-1)$ continually decreases.

Therefore, the difference between $a^m a^n$ and $a^{\frac{p}{q}+\frac{r}{s}}$ continually decreases; the difference between a^{m+n} and $a^{\frac{p}{q}+\frac{r}{s}}$ continually decreases; and each difference becomes as small as we please.

But, however great q and s may be, the expressions $a^m a^n$ and a^{m+n} have the same approximate value, $a^{\frac{p}{q}+\frac{r}{s}}$.

Therefore, as in § 204, we must have

$$a^m a^n = a^{m+n}.$$

The foregoing proof is easily extended to the case in which m and n are one or both negative.

Having proved for incommensurable exponents that $a^m a^n = a^{m+n}$, it is easily proved that:

$$\frac{a^m}{a^n} = a^{m-n};\ (a^m)^n = a^{mn};\ \sqrt[n]{a^m} = a^{\frac{m}{n}};\ a^m b^m = (ab)^m.$$

265. Properties of Logarithms. Let a be the base, M and N any positive numbers, m and n their logarithms to the base a; so that $a^m = M$, $a^n = N$,

$$m = \log_a M, \qquad n = \log_a N.$$

Then, in any system of logarithms:

(1) *The logarithm of* 1 *is* 0.

For, $a^0 = 1$. $\therefore\ 0 = \log_a 1$.

(2) *The logarithm of the base itself is* 1.

For, $a^1 = a$. $\therefore\ 1 = \log_a a$.

(3) *The logarithm of the reciprocal of a positive number is the negative of the logarithm of the number.*

For, if $a^n = N$, then $\dfrac{1}{N} = \dfrac{1}{a^n} = a^{-n}$.

$$\therefore\ \log_a\left(\frac{1}{N}\right) = -n = -\log_a N.$$

(4) *The logarithm of the product of two or more positive numbers is found by adding together the logarithms of the several factors.*

For, $M \times N = a^m \times a^n = a^{m+n}$.

$$\therefore\ \log_a (M \times N) = m + n = \log_a M + \log_a N.$$

Similarly for the product of three or more factors.

(5) *The logarithm of the quotient of two positive numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.*

For, $\dfrac{M}{N} = \dfrac{a^m}{a^n} = a^{m-n}$.

$$\therefore\ \log_a\left(\frac{M}{N}\right) = m - n = \log_a M - \log_a N.$$

(6) *The logarithm of a power of a positive number is found by multiplying the logarithm of the number by the exponent of the power.*

For, $$N^p = (a^n)^p = a^{np}.$$

$$\therefore \log_a(N^p) = np = p \log_a N.$$

(7) *The logarithm of the real positive value of a root of a positive number is found by dividing the logarithm of the number by the index of the root.*

For. $$\sqrt[r]{N} = \sqrt[r]{a^n} = a^{\frac{n}{r}}.$$

$$\therefore \log_a \sqrt[r]{N} = \frac{n}{r} = \frac{\log_a N}{r}.$$

266. In a system with a positive base greater than 1 the logarithms of all numbers greater than 1 are positive, and the logarithms of all positive numbers less than 1 are negative.

Conversely, in a system with a positive base less than 1 the logarithms of all numbers greater than 1 are negative, and the logarithms of all positive numbers less than 1 are positive.

267. Two Important Systems. Although the number of different systems of logarithms is unlimited, there are but two systems which are in common use. These are:

(1) The common system, also called the Briggs, denary, or decimal system, of which the base is 10.

(2) The natural system of which the base is the *natural base.*

The natural base, generally represented by e, is the fixed value which the sum of the series

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots\cdots$$

approaches as the number of terms is indefinitely increased; the value of e, carried to seven places of decimals, is 2.7182818.....

The common system is the system used in actual calculation; the natural system is used in the higher mathematics.

268. Common Logarithms. By logarithm in sections 268–282 is meant the common logarithm.

Since

$$10^0 = 1, \qquad 10^{-1}(=\tfrac{1}{10}) = 0.1,$$
$$10^1 = 10, \qquad 10^{-2}(=\tfrac{1}{100}) = 0.01,$$
$$10^2 = 100, \qquad 10^{-3}(=\tfrac{1}{1000}) = 0.001,$$

therefore

$$\log 1 = 0, \qquad \log 0.1 = -1,$$
$$\log 10 = 1, \qquad \log 0.01 = -2,$$
$$\log 100 = 2, \qquad \log 0.001 = -3.$$

Also, it is evident that the common logarithms of all numbers between

1 and	10	will be	$0 +$ a fraction,
10 and	100	will be	$1 +$ a fraction,
100 and	1000	will be	$2 +$ a fraction,
1 and	0.1	will be	$-1 +$ a fraction,
0.1 and	0.01	will be	$-2 +$ a fraction,
0.01 and	0.001	will be	$-3 +$ a fraction.

269. With common logarithms the mantissa is always made *positive*. Hence, in the case of numbers less than 1 whose logarithms are *negative*, the logarithm is made to consist of a *negative* characteristic and a *positive* mantissa.

When a logarithm consists of a *negative* characteristic and a *positive* mantissa, it is usual to write the minus sign *over* the characteristic, or else to add 10 to the characteristic and to indicate the subtraction of 10 from the resulting logarithm.

Thus, $\log 0.2 = \bar{1}.3010$, and this may be written $9.3010 - 10$.

270. The *characteristic* of the common logarithm of an integral number, or of a mixed number, is *one less* than the number of integral digits.

Thus, from § 268, $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$. Hence, the common logarithms of all numbers from 1 to 10 (that is, of all numbers consisting of *one* integral digit) will have 0 for characteristic; and the common logarithms of all numbers from 10 to 100 (that is, of all numbers consisting of *two* integral digits) will have 1 for characteristic; and so on, the characteristic increasing by one for each increase in the number of digits, and therefore being always one less than the number of digits.

271. The *characteristic* of the common logarithm of a decimal fraction is *negative*, and is equal to the number of the place occupied by the first significant figure of the decimal.

Thus, from § 268, $\log 0.1 = -1$, $\log 0.01 = -2$, $\log 0.001 = -3$. Hence, the common logarithms of all numbers from 0.1 to 1 will have -1 for a characteristic (the mantissa being *plus*); the common logarithms of all numbers from 0.01 to 0.1 will have -2 for a characteristic; the common logarithms of all numbers from 0.001 to 0.01 will have -3 for a characteristic; and so on, the characteristic always being *negative and equal to the number of the place occupied by the first significant figure of the decimal.*

272. The *mantissa* of the common logarithm of any integral number, or decimal fraction, depends only upon the digits of the number, and is unchanged so long as the *sequence of the digits* remains the same.

For, changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its common logarithm, therefore, will be increased or diminished by the *exponent* of that power of 10; and, since this exponent is *integral*, the *mantissa*, or decimal part of the logarithm, will be unaffected.

Thus,

$$27196 = 10^{4.4345}, \qquad 2.7196 = 10^{0.4345},$$
$$2719.6 = 10^{3.4345}, \qquad 0.27196 = 10^{9.4345-10},$$
$$27.196 = 10^{1.4345}, \qquad 0.0027196 = 10^{7.4345-10}.$$

One advantage of using the number *ten* as the base of a system of logarithms consists in the fact that the *mantissa* depends only on the *sequence of digits*, and the *characteristic* on the *position of the decimal point.*

273. In simplifying the logarithm of a root the equal positive and negative numbers to be added to the logarithm should be such that the resulting negative number, when divided by the index of the root, gives a quotient of -10.

Thus, $\log 0.002^{\frac{1}{3}} = \frac{1}{3}$ of $(7.3010 - 10)$.

The expression $\frac{1}{3}$ of $(7.3010 - 10)$

may be put in the form $\frac{1}{3}$ of $(27.3010 - 30)$, which is $9.1003 - 10$, since the addition of 20 to the 7, and of -20 to the -10, produces no change in the *value* of the logarithm.

Exercise 42.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the common logarithms of the following numbers by resolving the numbers into factors, and taking the sum of the logarithms of the factors:

1. log 6.	**5.** log 25.	**9.** log 0.021.	**13.** log 2.1.
2. log 15.	**6.** log 30.	**10.** log 0.35.	**14.** log 16.
3. log 21.	**7.** log 42.	**11.** log 0.0035.	**15.** log 0.056.
4. log 14.	**8.** log 420.	**12.** log 0.004.	**16.** log 0.63.

Find the common logarithms of the following:

17. 2^3.	**20.** 5^5.	**23.** $5^{\frac{1}{5}}$.	**26.** $7^{\frac{2}{7}}$.	**29.** $5^{\frac{7}{2}}$.
18. 5^2.	**21.** $2^{\frac{1}{3}}$.	**24.** $7^{\frac{1}{11}}$.	**27.** $5^{\frac{5}{3}}$.	**30.** $2^{\frac{11}{7}}$.
19. 7^4.	**22.** $5^{\frac{1}{2}}$.	**25.** $2^{\frac{3}{4}}$.	**28.** $3^{\frac{9}{11}}$.	**31.** $5^{\frac{3}{4}}$.

Find the common logarithms of the following:

32. $\frac{2}{5}$.	**36.** $\frac{5}{3}$.	**40.** $\frac{0.05}{3}$.	**44.** $\frac{0.05}{0.003}$.	**48.** $\frac{0.02^2}{3^3}$.
33. $\frac{2}{7}$.	**37.** $\frac{5}{2}$.	**41.** $\frac{0.005}{2}$.	**45.** $\frac{0.007}{0.02}$.	**49.** $\frac{3^3}{0.02^2}$.
34. $\frac{3}{5}$.	**38.** $\frac{7}{3}$.	**42.** $\frac{0.07}{5}$.	**46.** $\frac{0.02}{0.007}$.	**50.** $\frac{7^3}{0.02^2}$.
35. $\frac{3}{7}$.	**39.** $\frac{7}{2}$.	**43.** $\frac{5}{0.07}$.	**47.** $\frac{0.005}{0.07}$.	**51.** $\frac{0.07^3}{0.003^2}$.

274. The remainder obtained by subtracting the logarithm of a number from 10 is called the **cologarithm** of the number, or **arithmetical complement** of the logarithm of the number.

The cologarithm is abbreviated colog, and is most easily found by beginning with the characteristic of the logarithm and subtracting each figure from 9 down to the last significant figure, and subtracting that figure from 10.

Thus, $\log 7 = 0.8451$; and $\text{colog } 7 = 9.1549$. Colog 7 is readily found by subtracting, mentally, 0 from 9, 8 from 9, 4 from 9, 5 from 9, 1 from 10, and writing the resulting figure at each step.

275. If 10 be subtracted from the cologarithm of a number, the result is the logarithm of the reciprocal of that number.

For,
$$\begin{aligned}\log\frac{1}{N} &= \log 1 - \log N,\\ &= 0 - \log N,\\ &= (10 - \log N) - 10,\\ &= \text{colog } N - 10.\end{aligned}$$

276. The addition of a cologarithm -10 is equivalent to the subtraction of a logarithm.

For,
$$\begin{aligned}\text{colog } N - 10 &= (10 - \log N) - 10,\\ &= -\log N.\end{aligned}$$

277. The logarithm of a quotient may be found by *adding* together the *logarithm* of the dividend and the *cologarithm* of the divisor, and subtracting 10 from the result.

In finding a cologarithm when the *characteristic* of the logarithm is a *negative* number, it must be observed that the *subtraction* of a *negative* number is equivalent to the *addition* of an *equal positive* number.

Thus,
$$\begin{aligned}\log\frac{5}{0.002} &= \log 5 + \operatorname{colog} 0.002 - 10,\\ &= 0.6990 + 12.6990 - 10,\\ &= 3.3980.\end{aligned}$$

Here $\log 0.002 = \bar{3}.3010$, and in subtracting -3 from 9 the result is the same as adding $+3$ to 9.

Again,
$$\begin{aligned}\log\frac{2}{0.07} &= \log 2 + \operatorname{colog} 0.07 - 10,\\ &= 0.3010 + 11.1549 - 10,\\ &= 1.4559.\end{aligned}$$

Also,
$$\begin{aligned}\log\frac{0.07}{2^3} &= 8.8451 - 10 + 9.0970 - 10,\\ &= 17.9421 - 20,\\ &= 7.9421 - 10.\end{aligned}$$

Here, $\log 2^3 = 3 \log 2 = 3 \times 0.3010 = 0.9030.$
Hence, $\operatorname{colog} 2^3 = 10 - 0.9030 = 9.0970.$

278. Tables. A table of *four-place* common logarithms is here given, which contains the common logarithms of all numbers under 1000, *the decimal point and characteristic being omitted.* The logarithms of single digits 1, 8, etc., will be found at 10, 80, etc.

Tables containing logarithms of more places can be procured, but this table will serve for many practical uses, and will enable the student to use tables of five-place, seven-place, and ten-place logarithms, in work that requires greater accuracy.

In working with a four-place table, the numbers corresponding to the logarithms, that is, the *antilogarithms*, as they are called, may be carried to *four significant digits*.

279. To Find the Logarithm of a Number in this Table.

(1) Suppose it is required to find the logarithm of 65.7. In the column headed "N" look for the first two significant figures, and at the top of the table for the third significant figure. In the line with 65, and in the column headed 7, is seen 8176. To this number prefix the characteristic and insert the decimal point. Thus,

$$\log 65.7 = 1.8176.$$

(2) Suppose it is required to find the logarithm of 20347. In the line with 20, and in the column headed 3, is seen 3075; also in the line with 20, and in the 4 column, is seen 3096, and the difference between these two is 21. The difference between 20300 and 20400 is 100, and the difference between 20300 and 20347 is 47. Hence, $\frac{47}{100}$ of $21 = 10$, nearly, must be added to 3075. That is,

$$\log 20347 = 4.3085.$$

(3) Suppose it is required to find the logarithm of 0.0005076. In the line with 50, and in the 7 column, is seen 7050; in the 8 column, 7059: the difference is 9. The difference between 5070 and 5080 is 10, and the difference between 5070 and 5076 is 6. Hence, $\frac{6}{10}$ of $9 = 5$ must be added to 7050. That is,

$$\log 0.0005076 = 6.7055 - 10.$$

280. To Find a Number when its Logarithm is Given.

(1) Suppose it is required to find the number of which the logarithm is 1.9736.

Look for 9736 in the table. In the column headed "N," and in the line with 9736, is seen 94, and at the head of the column in which 9736 stands is seen 1. Therefore, write 941, and insert the decimal point as the characteristic directs. That is, the number required is 94.1.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

(2) Suppose it is required to find the number of which the logarithm is 3.7936.

Look for 7936 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 7931 and 7938; their difference is 7, and the difference between 7931 and 7936 is 5. Therefore, $\frac{5}{7}$ of the difference between the numbers corresponding to the mantissas, 7931 and 7938, must be added to the number corresponding to the mantissa 7931.

The number corresponding to the mantissa 7938 is 6220.

The number corresponding to the mantissa 7931 is 6210.

The difference between these numbers is 10,

and $$6210 + \tfrac{5}{7} \text{ of } 10 = 6217.$$

Therefore, the number required is 6217.

(3) Suppose it is required to find the number of which the logarithm is $7.3882 - 10$.

Look for 3882 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 3874 and 3892; their difference is 18, and the difference between 3874 and 3882 is 8. Therefore, $\frac{8}{18}$ of the difference between the numbers corresponding to the mantissas, 3874 and 3892, must be added to the number corresponding to the mantissa 3874.

The number corresponding to the mantissa 3892 is 2450.

The number corresponding to the mantissa 3874 is 2440.

The difference between these numbers is 10,

and $$2440 + \tfrac{8}{18} \text{ of } 10 = 2444.$$

Therefore, the number required is 0.002444.

Exercise 43.

Find, from the table, the common logarithms of:

1. 60.	**4.** 3780.	**7.** 70633.	**10.** 0.0004523.
2. 101.	**5.** 5432.	**8.** 12028.	**11.** 0.01342.
3. 999.	**6.** 9081.	**9.** 0.00987.	**12.** 0.19873.

Find antilogarithms to the following common logarithms:

13. 4.2488.	**15.** 4.7317.	**17.** 9.0410 — 10.
14. 3.6330.	**16.** 1.9730.	**18.** 9.8420 — 10.

281. Examples.

(1) Find the product of $908.4 \times 0.05392 \times 2.117$.

$$\begin{aligned} \log\ 908.4 &= 2.9583 \\ \log 0.05392 &= 8.7318 - 10 \\ \log\ 2.117 &= \underline{0.3257\qquad\ } \\ & \ 2.0158 = \log 103.7. \quad \textit{Ans.} \end{aligned}$$

When any of the factors are *negative*, find their logarithms without regard to the signs; write — after the logarithm that corresponds to a negative number. If the number of logarithms so marked be *odd*, the product is *negative*; if *even*, the product is *positive*.

(2) Find the product of $4.52 \times (-0.3721) \times 0.912$.

$$\begin{aligned} \log\ 4.52 &= 0.6551 \qquad\quad + \\ \log 0.3721 &= 9.5706 - 10 - \\ \log\ 0.912 &= \underline{9.9600 - 10 +} \\ & \ 0.1857 = \log -1.534. \quad \textit{Ans.} \end{aligned}$$

(3) Find the quotient of $\dfrac{8.3709 \times 834.637}{7308.946}$.

$$\begin{aligned} \log\ 8.3709 &= 0.9227 \\ \log\ 834.637 &= 2.9215 \\ \text{colog } 7308.946 &= \underline{6.1362 - 10} \\ & \ 9.9804 - 10 = \log 0.9558. \quad \textit{Ans.} \end{aligned}$$

(4) Find the cube of 0.0497.

$$\log 0.0497 = 8.6964 - 10$$
$$\underline{\quad 3 \quad}$$
$$6.0892 - 10 = \log 0.0001228. \quad \textit{Ans.}$$

(5) Find the fourth root of 0.00862.

$$\log 0.00862 = 7.9355 - 10$$
$$\underline{30. \qquad -30}$$
$$4)\overline{37.9355 - 40}$$
$$9.4839 - 10 = \log 0.3047. \quad \textit{Ans.}$$

282. An **exponential equation**, that is, an equation in which the exponent involves the unknown number, is easily solved by logarithms.

Find the value of x in $81^x = 10$.

$$81^x = 10.$$
$$\therefore \log (81^x) = \log 10,$$
$$x \log 81 = \log 10,$$
$$x = \frac{\log 10}{\log 81} = \frac{1.0000}{1.9085} = 0.524. \quad \textit{Ans.}$$

Exercise 44.

Find by logarithms the following products:

1. 948.76×0.043875.
2. 3.4097×0.0087634.
3. 830.75×0.0003769.
4. 8.4395×0.98274.
5. $7564 \times (-0.003764)$.
6. $3.7648 \times (-0.083497)$.
7. $-5.840359 \times (-0.00178)$.
8. -8945.07×73.846.

Find by logarithms:

9. $\dfrac{70654}{54013}$.
10. $\dfrac{7.652}{-0.06875}$.
11. $\dfrac{0.07654}{83.947 \times 0.8395}$.
12. $\dfrac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.

13. 1.1768^5.	**16.** $\left(\frac{73}{61}\right)^{11}$.	**19.** $\left(\frac{951}{823}\right)^6$.	**22.** $\left(8\frac{3}{4}\right)^{2.3}$.
14. 1.3178^{10}.	**17.** $\left(\frac{14}{51}\right)^7$.	**20.** $\left(7\frac{6}{11}\right)^{0.38}$.	**23.** $\left(5\frac{81}{87}\right)^{0.375}$.
15. $11^{\frac{1}{5}}$.	**18.** $906.80^{\frac{1}{4}}$.	**21.** $2.5637^{\frac{3}{11}}$.	**24.** $\left(9\frac{20}{48}\right)^{\frac{1}{5}}$.

25. $\sqrt[5]{\dfrac{0.0075433^2 \times 78.343 \times 8172.4^{\frac{1}{3}} \times 0.00052}{64285^{\frac{1}{3}} \times 154.27^4 \times 0.001 \times 586.79^{\frac{1}{2}}}}$.

26. $\sqrt[7]{\dfrac{0.03271^2 \times 53.429 \times 0.77542^3}{32.769 \times 0.000371^4}}$.

27. $\sqrt[3]{\dfrac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}$.

Find x from the equations:

28. $5^x = 12$.	**30.** $7^x = 25$.	**32.** $(0.4)^{-x} = 7$.
29. $4^x = 40$.	**31.** $(1.3)^x = 7.2$.	**33.** $(0.9)^{\frac{1}{x^2}} = (4.7)^{-\frac{1}{3}}$.

283. Change of System. Logarithms to any base a may be converted into logarithms to any other base b as follows:

Let N be any number, and let

$$n = \log_a N \text{ and } m = \log_b N.$$

Then, $N = a^n$ and $N = b^m$.

$$\therefore\ a^n = b^m.$$

Taking logarithms to any base whatever,

$$n \log a = m \log b, \qquad \S\ 265$$

or, $\log a \times \log_a N = \log b \times \log_b N$,

from which $\log_b N$ may be found when $\log a$, $\log b$, and $\log_a N$ are given; and conversely, $\log_a N$ may be found when $\log a$, $\log b$, and $\log_b N$ are given.

284. If $a = 10$, $b = e$, and $N = 10$, we have (§ 283)

$$\log_{10} 10 \times \log_{10} 10 = \log_{10} e \times \log_e 10.$$

$$\therefore \log_e 10 = \frac{1}{\log_{10} e}.$$

From tables $$\log_{10} e = 0.4342945.$$

$$\therefore \log_e 10 = 2.3025851.$$

285. If $a = 10$, $b = e$, and N is any number,

$$\log_{10} 10 \times \log_{10} N = \log_{10} e \times \log_e N. \qquad \text{§ 283}$$

$$\therefore \log_e N = \frac{1}{\log_{10} e} \times \log_{10} N,$$

and $$\log_{10} N = \log_{10} e \times \log_e N.$$

Hence, to convert common into natural logarithms, multiply by 2.3025851; and to convert natural into common logarithms multiply by 0.4342945.

Exercise 45.

Find to four digits the natural logarithms of:

1. 2.	**3.** 100.	**5.** 7.89.	**7.** 2.001.
2. 3.	**4.** 32.5.	**6.** 1.23.	**8.** 0.0931.

Find to four digits:

9. $\log_2 7$.	**11.** $\log_4 9$.	**13.** $\log_9 8$.	**15.** $\log_7 14$.
10. $\log_3 4$.	**12.** $\log_5 7$.	**14.** $\log_8 5$.	**16.** $\log_5 102$.

17. Find the logarithm of 4 in the system of which $\frac{1}{3}$ is the base.

18. Find the logarithm of $\frac{7}{11}$ in the system of which 0.5 is the base.

19. Find the base of the system in which the logarithm of 8 is $\frac{2}{3}$.

20. Find the base of the system in which the logarithm of $\frac{2}{3}$ is $-\frac{3}{5}$.

CHAPTER XX.

INTEREST AND ANNUITIES.

286. Simple Interest.

If the principal be represented by P,
the interest on \$1 for one year by r,
the amount of \$1 for one year by R,
the number of years by n,
the amount of P for n years by A,

Then $R = 1 + r$.

Simple interest on P for a year $= Pr$,
Amount of P for a year $= PR$,
Simple interest on P for n years $= Pnr$,
Amount of P for n years $= P(1 + nr)$,
that is, $A = P(1 + nr)$.

287. When any three of the quantities A, P, n, r are given, the fourth may be found.

Required the rate when \$500 in 4 years at simple interest amounts to \$610.

r is required; A, P, n are given.

$$A = P(1 + nr),$$

or

$$A = P + Pnr.$$

$$\therefore Pnr = A - P,$$

$$\therefore r = \frac{A - P}{Pn} = \frac{610 - 500}{2000} = 0.055.$$

$5\frac{1}{2}$ per cent. *Ans.*

288. Since P will in n years amount to A, it is evident that P at the present time may be considered equivalent in value to A due at the end of n years; so that P may be regarded as the *present worth* of a given future sum A.

Find the present worth of \$600, due in 2 years, the rate of interest being 6 per cent.

$$A = P(1+nr).$$

$$\therefore\ P = \frac{A}{1+nr} = \frac{\$600}{1+0.12} = \$535.71.$$

289. Compound Interest.

I. When compound interest is reckoned payable *annually*,

The amount of P dollars in

$$\begin{aligned} &1 \text{ year is } && P(1+r) \text{ or } PR, \\ &2 \text{ years is } && PR(1+r) \text{ or } PR^2, \\ &n \text{ years is } && PR^n. \end{aligned}$$

That is, $$A = PR^n.$$

Hence, also, $$P = \frac{A}{R^n}.$$

II. When compound interest is payable *semi-annually*,

The amount of P dollars in

$$\tfrac{1}{2} \text{ year is } P\left(1+\frac{r}{2}\right),$$

$$1 \text{ year is } P\left(1+\frac{r}{2}\right)^2,$$

$$n \text{ years is } P\left(1+\frac{r}{2}\right)^{2n}.$$

That is, $$A = P\left(1+\frac{r}{2}\right)^{2n}.$$

III. When the interest is payable *quarterly*,

$$A = P\left(1+\frac{r}{4}\right)^{4n}.$$

IV. When the interest is payable *monthly*,

$$A = P\left(1 + \frac{r}{12}\right)^{12n}.$$

V. When interest is payable q times a year,

$$A = P\left(1 + \frac{r}{q}\right)^{qn}.$$

Find the present worth of \$500, due in 4 years, at 5 per cent compound interest.

$$A = P(1+r)^4.$$

$$\therefore\ P = \frac{A}{(1+r)^4} = \frac{\$500}{(1.05)^4} = \$411.36.\quad \textit{Ans.}$$

290. Sinking Funds. If the sum set apart at the end of each year to be put at compound interest be represented by S, then,

The sum at the end of the

$$\begin{aligned} &\text{first year} &&= S,\\ &\text{second year} &&= S + SR,\\ &\text{third year} &&= S + SR + SR^2,\\ &n\text{th year} &&= S + SR + SR^2 + \cdots\cdots + SR^{n-1}. \end{aligned}$$

That is, the amount $A = S + SR + SR^2 + \cdots\cdots + SR^{n-1}$.

$$\therefore\ AR = SR + SR^2 + SR^3 + \cdots\cdots + SR^n.$$

$$\therefore\ AR - A = SR^n - S.$$

$$\therefore\ A = \frac{S(R^n - 1)}{R - 1},$$

or,

$$A = \frac{S(R^n - 1)}{r}.$$

(1) If \$10,000 be set apart annually, and put at 6 per cent compound interest for 10 years, what will be the amount?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$10{,}000(1.06^{10} - 1)}{0.06}.$$

By logarithms the amount is found to be \$131,740 (*nearly*).

(2) A county owes $60,000. What sum must be set apart annually, as a sinking fund, to cancel the debt in 10 years, provided money is worth 6 per cent?

$$S=\frac{Ar}{R^n-1}=\frac{\$60,000\times 0.06}{1.06^{10}-1}=\$4555\ (\textit{nearly}).$$

NOTE. The amount of tax required yearly is $3600 for the *interest* and $4555 for the sinking fund; that is, $8155.

291. Annuities. A sum of money that is payable yearly, or in parts at fixed periods in the year, is called an **annuity**.

To find the amount of an unpaid annuity when the interest, time, and rate per cent are given.

The sum due at the *end* of the

$$\begin{aligned} \text{first year} &= S,\\ \text{second year} &= S+SR,\\ \text{third year} &= S+SR+SR^2,\\ n\,th\ \text{year} &= S+SR+SR^2+\cdots\cdots+SR^{n-1}. \end{aligned}$$

That is, $$A=\frac{S(R^n-1)}{r}$$ § 290

An annuity of $1200 was unpaid for 6 years. What was the amount due if interest be reckoned at 6 per cent?

$$A=\frac{S(R^n-1)}{r}=\frac{\$1200(1.06^6-1)}{0.06}=\$8370.$$

292. *To find the present worth of an annuity when the time it is to continue and the rate per cent are given.*

Let P denote the present worth. Then the amount of P for n years will be equal to A the amount of the annuity for n years.

But the amount of P for n years

$$=P(1+r)^n=PR^n,$$ § 289

and $$A=\frac{S(R^n-1)}{R-1}.$$ § 291

$$\therefore\ PR^n=\frac{S(R^n-1)}{R-1}.$$

$$\therefore\ P=\frac{S}{R^n}\times\frac{R^n-1}{R-1}.$$

This equation may be written

$$P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}=\frac{S}{R-1}\left(1-\frac{1}{R^n}\right).$$

As n increases, the expression

$$\left(1-\frac{1}{R^n}\right)$$

approaches 1. Therefore if the annuity be *perpetual*,

$$P=\frac{S}{R-1}=\frac{S}{r}.$$

(1) Find the present worth of an annual pension of \$105, for 5 years, at 4 per cent interest.

$$P=\frac{S}{R^n}\times\frac{R^n-1}{R-1}$$

$$=\frac{\$105}{1.04^5}\times\frac{1.04^5-1}{1.04-1}=\$467\ (\textit{nearly}).$$

(2) Find the present worth of a perpetual scholarship that pays \$300 annually, at 6 per cent interest.

$$P=\frac{S}{r}=\frac{\$300}{0.06}=\$5000.$$

293. *To find the present worth of an annuity that begins in a given number of years, when the time it is to continue and the rate per cent are given.*

Let p denote the number of years before the annuity begins, and q the number of years the annuity is to continue.

Then the present worth of the annuity to the time it *terminates* is

$$\frac{S}{R^{p+q}}\times\frac{R^{p+q}-1}{R-1},$$

and the present worth of the annuity to the time it *begins* is

$$\frac{S}{R^p} \times \frac{R^p - 1}{R - 1}.$$

Hence,

$$P = \left(\frac{S}{R^{p+q}} \times \frac{R^{p+q} - 1}{R - 1}\right) - \left(\frac{S}{R^p} \times \frac{R^p - 1}{R - 1}\right).$$

$$\therefore\ P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1}.$$

If the annuity is to begin at the end of p years, and to be perpetual, the formula

$$P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1}$$

becomes
$$P = \frac{S}{R^p(R - 1)} \times \frac{R^q - 1}{R^q}.$$

And since $\dfrac{R^q - 1}{R^q}$ approaches 1 (§ 292),

$$P = \frac{S}{R^p(R - 1)}.$$

(1) Find the present worth of an annuity of \$5000, to begin in 6 years, and to continue 12 years, at 6 per cent interest.

$$P = \frac{S}{R^{p+q}} \times \frac{R^q - 1}{R - 1}$$

$$= \frac{\$5000}{1.06^{18}} \times \frac{1.06^{12} - 1}{0.06} = \$29{,}550.$$

(2) Find the present worth of a perpetual annuity of \$1000, to begin in 3 years, at 4 per cent interest.

$$P = \frac{S}{R^p(R - 1)} = \frac{\$1000}{1.04^3 \times 0.04} = \$22{,}225.$$

294. *To find the annuity when the present worth, the time, and the rate per cent are given.*

$$P = \frac{S(R^n - 1)}{R^n(R-1)}. \qquad \S\ 292$$

$$\therefore\ S = \frac{PR^n(R-1)}{R^n - 1} = Pr \times \frac{R^n}{R^n - 1}.$$

What annuity for 5 years will \$4675 give when interest is reckoned at 4 per cent?

$$S = Pr \times \frac{R^n}{R^n - 1} = \$4675 \times 0.04 \times \frac{1.04^5}{1.04^5 - 1} = \$1050.$$

295. Life Insurance. In order that a certain sum may be secured, to be payable at the death of a person, he pays yearly a fixed *premium*.

If P denote the premium to be paid for n years to insure an amount A, to be paid immediately after the last premium, then

$$A = \frac{P(R^n - 1)}{R - 1}. \qquad \S\ 290$$

$$\therefore\ P = \frac{A(R-1)}{R^n - 1} = \frac{Ar}{R^n - 1}.$$

If A is to be paid a year after the last premium, then

$$P = \frac{A(R-1)}{R(R^n - 1)} = \frac{Ar}{R(R^n - 1)}.$$

NOTE. In the calculation of life insurances it is necessary to employ tables which show for any age the probable duration of life.

296. Bonds. If P denote the price of a bond that has n years to run, and bears r per cent interest, S the face of the bond, and q the current rate of interest, what interest on his investment will a purchaser of such a bond receive?

Let x denote the rate of interest on the investment.

Then $P(1+x)^n$ is the value of the purchase money at the end of n years.

$Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots\cdots+Sr+S$ is the amount of money received on the bond if the interest received from the bond is put immediately at compound interest at q per cent.

$$\text{But}\quad Sr(1+q)^{n-1}+Sr(1+q)^{n-2}+\cdots\cdots+Sr+S$$
$$=S+\frac{Sr[(1+q)^n-1]}{q}.$$
$$\therefore\ P(1+x)^n=S+\frac{Sr[(1+q)^n-1]}{q}.$$
$$\therefore\ 1+x=\left(\frac{S}{P}+\frac{Sr[(1+q)^n-1]}{Pq}\right)^{\frac{1}{n}}$$
$$=\left(\frac{Sq+Sr(1+q)^n-Sr}{Pq}\right)^{\frac{1}{n}}.$$

(1) What interest will a person receive on his investment if he buys at 114 a 4 per cent bond that has 26 years to run, money being worth $3\frac{1}{2}$ per cent?

$$1+x=\left(\frac{3.5+4(1.035)^{26}-4}{3.99}\right)^{\frac{1}{26}}.$$

By logarithms, $1+x=1.033$.

That is, the purchaser will receive $3\frac{1}{3}$ per cent for his money.

(2) At what price must 7 per cent bonds, running 12 years, with the interest payable semi-annually, be bought, in order that the purchaser may receive on his investment 5 per cent, interest semi-annual, which is the current rate of interest?

$$P(1+x)^n=\frac{Sq+Sr(1+q)^n-Sr}{q}.$$
$$\therefore\ P=\frac{Sq+Sr(1+q)^n-Sr}{q(1+x)^n}.$$

In this case $S=100$; and, as the interest is semi-annual,

$$q=0.025,\quad r=0.035,\quad n=24,\quad x=0.025.$$

Hence, $$P=\frac{2.5+3.5(1.025)^{24}-3.5}{0.025(1.025)^{24}}.$$

By logarithms, $P=118$.

Exercise 46.

1. In how many years will \$100 amount to \$1050, at 5 per cent compound interest?

2. In how many years will \$$A$ amount to \$$B$ (1) at simple interest, (2) at compound interest, r and R being used in their usual sense?

3. Find the difference (to five places of decimals) between the amount of \$1 in 2 years, at 6 per cent compound interest, according as the interest is due yearly or monthly.

4. At 5 per cent, find the amount of an annuity of \$$A$ which has been left unpaid for 4 years.

5. Find the present value of an annuity of \$100 for 5 years, reckoning interest at 4 per cent.

6. A perpetual annuity of \$1000 is to be purchased, to begin at the end of 10 years. If interest is reckoned at 3½ per cent, what should be paid for it?

7. A debt of \$1850 is discharged by two payments of \$1000 each, at the end of one and two years. Find the rate of interest paid.

8. Reckoning interest at 4 per cent, what annual premium should be paid for 30 years, in order to secure \$2000 to be paid at the end of that time, the premium being due at the beginning of each year?

9. An annual premium of \$150 is paid to a life-insurance company for insuring \$5000. If money is worth 4 per cent, for how many years must the premium be paid in order that the company may sustain no loss?

10. What may be paid for bonds due in 10 years, and bearing semi-annual coupons of 4 per cent each, in order to realize 3 per cent semi-annually, if money is worth 3 per cent semi-annually?

11. When money is worth 2 per cent semi-annually, if bonds having 12 years to run, and bearing semi-annual coupons of $3\frac{1}{2}$ per cent each, are bought at $114\frac{1}{8}$, what per cent is realized on the investment?

12. If $126 is paid for bonds due in 12 years, and yielding $3\frac{1}{2}$ per cent semi-annually, what per cent is realized on the investment, provided money is worth 2 per cent semi-annually?

13. A person borrows $600.25. How much must he pay annually that the whole debt may be discharged in 35 years, interest being reckoned at 4 per cent?

14. A perpetual annuity of $100 a year is sold for $2500. At what rate is the interest reckoned?

15. A perpetual annuity of $320, to begin 10 years hence, is to be purchased. If interest is reckoned at $3\frac{1}{5}$ per cent, what should be paid for it?

16. A sum of $10,000 is loaned at 4 per cent. At the end of the first year a payment of $400 is made; and at the end of each following year a payment is made greater by 30 per cent than the preceding payment. Find in how many years the debt will be paid.

17. A man with a capital of $100,000 spends every year $9000. If the current rate of interest is 5 per cent, in how many years will he be ruined?

18. Find the amount of $365 at compound interest for 20 years, at 5 per cent.

CHAPTER XXI.

CHOICE.

297. Fundamental Principle. *If one thing can be done in* a *different ways, and, when it has been done, a second thing can be done in* b *different ways, then the two things can be done together in* a $\times$ b *different ways.*

For, corresponding to the *first* way of doing the first thing, there are b different ways of doing the second thing; corresponding to the *second* way of doing the first thing, there are b different ways of doing the second thing; and so on for *each* of the a different ways of doing the first thing. Therefore there are $a \times b$ different ways of doing the two things together.

(1) If a box contains four capital letters, A, B, C, D, and three small letters, x, y, z, in how many different ways may two letters, one a capital letter and one a small letter, be selected?

A capital letter may be selected in four different ways, since any one of the letters A, B, C, D, may be selected A small letter may be selected in three different ways, since any one of the letters x, y, z, may be selected. Any small letter may be put with any capital letter.

Thus, with A we may put x, or y, or z;
with B we may put x, or y, or z;
with C we may put x, or y, or z;
with D we may put x, or y, or z.

Hence the number of ways in which a selection may be made is 4×3, or 12. These ways are:

Ax	Bx	Cx	Dx
Ay	By	Cy	Dy
Az	Bz	Cz	Dz

(2) On a shelf are 7 English, 5 French, and 9 German books. In how many ways may two books, not in the same language, be selected?

An English book and a French book can be selected in 7×5, or 35, ways. A French book and a German book in 5×9, or 45, ways. An English book and a German book in 7×9, or 63, ways.

Hence, there is a choice of $35 + 45 + 63$, or 143, ways. *Ans.*

(3) Out of the ten figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, how many numbers, each consisting of two figures, can be formed?

Since 0 has no value in the left-hand place, the left-hand place can be filled in 9 ways.

The right-hand place can be filled in 10 ways, since *repetitions* of the digits are allowed (as 22, 33, etc.).

Hence, the whole number is 9×10, or 90. *Ans.*

298. By successive application of the principle of § 297 it may be shown that,

If one thing can be done in a *different ways, and then a second thing can be done in* b *different ways, then a third thing in* c *different ways, then a fourth thing in* d *different ways, etc., the number of different ways of doing all the things together will be* $a \times b \times c \times d$, *etc.*

For, the first and second things can be done together in $a \times b$ different ways (§ 297), and the third thing in c different ways; hence, by § 297, the first and second things and the third thing can be done together in $(a \times b) \times c$ different ways. Therefore, the first three things can be done in $a \times b \times c$ different ways. And so on for any number of things.

Ex. In how many ways can four Christmas presents be given to four boys, one to each boy?

The first present may be given to any one of the boys; hence there are 4 ways of disposing of it.

The second present may be given to any one of the other three boys; hence there are 3 ways of disposing of it.

The third present may be given to either of the other two boys; hence there are 2 ways of disposing of it.

The fourth present must be given to the last boy; hence there is only 1 way of disposing of it.

There are, then, $4 \times 3 \times 2 \times 1$, or 24, ways. *Ans.*

299. Selections and Arrangements.

(1) In how many ways can a vowel and a consonant be chosen out of the alphabet?

Since there are in the alphabet 6 vowels and 20 consonants, a vowel can be chosen in 6 ways and a consonant in 20 ways, and both (§ 297) in 6×20, or 120, ways.

(2) In how many ways can a two-lettered word be made, containing one vowel and one consonant?

The vowel can be chosen in 6 ways and the consonant in 20 ways; and then each combination of a vowel and a consonant can be written in 2 ways; as *ac*, *ca*.

Hence, the whole number of ways is $6 \times 20 \times 2$, or 240.

These two examples show the difference between a *selection* or *combination* of different things, and an *arrangement* or *permutation* of the same things.

Thus, *ac* form a selection of a vowel and a consonant, and *ac* and *ca* form two different *arrangements* of this selection.

From (1) it is seen that 120 different selections can be made with a vowel and a consonant; and from (2) it is seen that 240 different *arrangements* can be made with the same.

Again, *a*, *b*, *c* is a selection of three letters from the alphabet. This selection admits of 6 different arrangements, as follows:

abc	*bca*	*cab*
acb	*bac*	*cba*

A **selection** or **combination** of any number of things is a group of that number of things put together without regard to their order.

An **arrangement** or **permutation** of any number of things is a group of that number of things put together, regard being paid to their order.

300. Arrangements, Things all different. *The number of different arrangements (or permutations) of* **n** *different things taken all together is*

$$n(n-1)(n-2)(n-3) \cdots\cdots 3 \times 2 \times 1.$$

For, the first place can be filled in n ways, then the second place in $n-1$ ways, then the third place in $n-2$ ways, and so on to the last place, which can be filled in only 1 way.

Hence (§ 298) the whole number of arrangements is the continued product of all these numbers,

$$n(n-1)(n-2)(n-3) \cdots\cdots 3 \times 2 \times 1.$$

For the sake of brevity this product is written $\underline{|n}$, and is read **factorial n.**

Observe that $\quad 1 \times 2 \cdots\cdots (n-1)\, n = \underline{|n}$.

Ex. How many different arrangements of nine letters each can be formed with the letters in *Cambridge?*

There are nine letters. In making any arrangement any one of the letters can be put in the first place. Hence, the first place can be filled in 9 ways.

Then the second place can be filled with any one of the remaining eight letters; that is, in 8 ways.

In like manner, the third place can be filled in 7 ways, the fourth place in 6 ways, and so on; and, lastly, the ninth place in 1 way.

If the nine places be indicated by Roman numerals, the result is (§ 298) as follows:

$$\begin{array}{ccccccccl} \text{I.} & \text{II.} & \text{III.} & \text{IV.} & \text{V.} & \text{VI.} & \text{VII.} & \text{VIII.} & \text{IX.} \\ 9 \times & 8 \times & 7 \times & 6 \times & 5 \times & 4 \times & 3 \times & 2 \times & 1 = 362{,}880 \text{ ways.} \end{array}$$

Hence, there are 362,880 different arrangements possible.

301. *The number of different arrangements of* n *different things taken* r *at a time is*

$$n(n-1)(n-2) \cdots\cdots \text{ to } r \text{ factors,}$$

that is, $$n(n-1)(n-2) \cdots\cdots [n-(r-1)],$$

or $$n(n-1)(n-2) \cdots\cdots (n-r+1).$$

For, the first place can be filled in n ways, the second place in $n-1$ ways, the third place in $n-2$ ways, and the rth place in $n-(r-1)$ ways.

Let $P_{n,r}$ represent the number of arrangements of n different things taken r at a time. Then

$$P_{n,r} = n(n-1)(n-2) \cdots\cdots \text{ to } r \text{ factors.}$$
$$= n(n-1)(n-2) \cdots\cdots (n-r+1).$$

Ex. How many different arrangements of four letters each can be formed from the letters in *Cambridge ?*

There are nine letters and four places to be filled.

The first place can be filled in 9 ways. Then the second place can be filled in 8 ways. Then the third place in 7 ways, and the fourth place in 6 ways.

If the places be indicated by I., II., III., IV., the result is (§ 298)

I. II. III. IV.
$9 \times 8 \times 7 \times 6 = 3024$ ways.

Hence, there are 3024 different arrangements possible.

302. Selections, Things all different. *The number of different selections (or combinations) of* n *different things taken* r *at a time is*

$$\frac{n(n-1)(n-2) \cdots\cdots (n-r+1)}{\lfloor r}.$$

To prove this, let $C_{n,r}$ represent the number of different selections (or combinations) of n different things taken r at a time.

Take one selection of r things; from this selection $\lfloor r$ arrangements can be made (§ 300).

Take a second selection; from this selection $\underline{|r}$ arrangements can be made. And so on for *each* of the $C_{n,\,r}$ selections.

Hence, $C_{n,\,r} \times \underline{|r}$ is the number of *arrangements* of n different things taken r at a time; or

$$C_{n,\,r} \times \underline{|r} = P_{n,\,r}.$$

$$\therefore \; C_{n,\,r} = \frac{P_{n,\,r}}{\underline{|r}}.$$

$$\therefore \; C_{n,\,r} = \frac{n(n-1)(n-2) \cdots\cdots (n-r+1)}{\underline{|r}}.$$

Ex. In how many different ways can three vowels be selected from the five vowels a, e, i, o, u.

The number of different ways in which we can *arrange* 3 vowels out of 5 is (§ 301) $5 \times 4 \times 3$, or 60.

These 60 arrangements might be obtained by first forming all the possible selections of 3 vowels out of 5, and then arranging the 3 vowels in each selection in as many ways as possible.

Since each selection can be arranged in $\underline{|3}$, or 6, ways (§ 300), the number of selections is $\frac{60}{6}$ or 10. *Ans.*

The formula applied to this problem gives

$$C_{5,\,3} = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10.$$

303. Selections, Second Formula. Multiplying both numerator and denominator of the expression for the number of selections in the last example by 2×1, we have

$$C_{5,\,3} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 2 \times 1} = \frac{\underline{|5}}{\underline{|3}\,\underline{|2}}.$$

In general, multiplying both numerator and denominator of the expression for $C_{n,\,r}$ in § 302 by $\underline{|n-r}$, we have

$$C_{n,\,r} = \frac{n(n-1) \cdots\cdots (n-r+1)(n-r) \cdots\cdots 1}{\underline{|r} \times (n-r) \cdots\cdots 1}$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

This second form is more compact than the first, and is more easily remembered.

NOTE. In reducing a result expressed in the above form, it is to be observed that $\underline{|n-r}$ cancels all the factors of the numerator from 1 up to and including $n-r$. Thus, in $\frac{\underline{|12}}{\underline{|5}\,\underline{|7}}$, $\underline{|7}$ cancels all the factors of $\underline{|12}$ from 1 up to and including 7; so that

$$\frac{\underline{|12}}{\underline{|5}\,\underline{|7}}=\frac{12\times 11\times 10\times 9\times 8}{1\times 2\times 3\times 4\times 5}=792.$$

304. Theorem. *The number of selections of* n *things taken* r *at a time is the same as the number of selections of* n *things taken* n − r *at a time.*

$$\text{For, } C_{n,\,n-r}=\frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}}=\frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}=C_{n,\,r}.$$

This is also evident from the fact that for every selection of r things *taken*, a selection of $n-r$ things is *left*.

Thus, out of 8 things, 3 things can be selected in the same number of ways as 5 things; namely,

$$\frac{\underline{|8}}{\underline{|3}\,\underline{|5}}=\frac{8\times 7\times 6}{\underline{|3}}=56 \text{ ways.}$$

NOTE. Evidently $C_{1,\,1}=1$; also $C_{1,\,1}=\frac{\underline{|1}}{\underline{|1}\,\underline{|0}}=\frac{1}{\underline{|0}}$;

$$\therefore \frac{1}{\underline{|0}}=1, \text{ and } \underline{|0}=1.$$

305. Examples in Selections and Arrangements. Of the arrangements possible with the letters of the word *Cambridge*, taken all together.

(1) How many will begin with a vowel?

In filling the nine places of any arrangement the first place can be filled in only 3 ways, the other places in $\underline{|8}$ ways.

Hence, the answer is (§ 298)

$$3\times \underline{|8}=120{,}960.$$

(2) How many will both begin and end with a vowel?

The first place can be filled in 3 ways, the last place in 2 ways (one vowel having been used), and the remaining seven places in $\underline{|7}$ ways.

Hence, the answer is (§ 298)

$$3 \times 2 \times \underline{|7} = 30{,}240.$$

(3) How many will begin with *Cam?*

The answer is evidently $\underline{|6}$; since our only choice lies in arranging the remaining six letters of the word.

(4) How many will have the letters *c a m* standing together?

This may be resolved into arranging the group *c a m* and the last six letters, regarded as seven distinct elements, and then arranging the letters *c a m*.

The first can be done in $\underline{|7}$ ways, and the second in $\underline{|3}$ ways. Hence both can be done in $\underline{|7} \times \underline{|3} = 30{,}240$ ways. *Ans.*

In how many ways can the letters of the word *Cambridge* be written:

(5) Without changing the *place* of any vowel?

The second, sixth, and ninth places can be filled each in only 1 way; the other places in $\underline{|6}$ ways.

Therefore, the whole number of ways is $\underline{|6} = 720$. *Ans.*

(6) Without changing the *order* of the three vowels?

The vowels in the different arrangements are to be kept in the order *a*, *i*, *e*.

One of the six consonants can be placed in 4 ways: *before a*, *between a* and *i*, *between i* and *e*, and *after e*.

Then a second consonant can be placed in 5 ways, a third consonant in 6 ways, a fourth consonant in 7 ways, a fifth consonant in 8 ways, and the last consonant in 9 ways. Hence the whole number of ways is

$$4 \times 5 \times 6 \times 7 \times 8 \times 9, \text{ or } 60{,}480. \quad \textit{Ans.}$$

(7) Out of 20 consonants, in how many ways can 18 be selected?

The 18 can be selected in the same number of ways as 2; and the number of ways in which 2 can be selected is

$$\frac{20 \times 19}{2} = 190. \quad \textit{Ans.}$$

(8) In how many ways can the same choice be made so as always to include the letter b?

Taking b first, we must then select 17 out of the remaining 19 consonants. This can be done in

$$\frac{19 \times 18}{2} = 171 \text{ ways.} \quad \textit{Ans.}$$

(9) In how many ways can the same choice be made so as to include b and not to include c?

Taking b first, we have then to choose 17 out of 18, c being excluded. This can be done in 18 ways. *Ans.*

(10) From 20 Republicans and 6 Democrats, in how many ways can 5 different offices be filled, three of which must be filled by Republicans, and the other two by Democrats?

The first three offices can be assigned to 3 Republicans in

$$20 \times 19 \times 18 = 6840 \text{ ways;}$$

and the other two offices can be assigned to 2 Democrats in

$$6 \times 5 = 30 \text{ ways.}$$

There is, then, a choice of $6840 \times 30 = 205{,}200$ different ways.

(11) Out of 20 consonants and 6 vowels, in how many ways can we make a word consisting of 3 different consonants and 2 different vowels?

Three consonants can be selected in $\frac{20 \times 19 \times 18}{1 \times 2 \times 3} = 1140$ ways, and two vowels in $\frac{6 \times 5}{1 \times 2} = 15$ ways. Hence the 5 letters can be selected in $1140 \times 15 = 17{,}100$ ways.

When five letters have been so selected, they can be arranged in $\underline{|5} = 120$ different orders. Hence, there are $17{,}100 \times 120 = 2{,}052{,}000$ different ways of making the word.

Observe that the letters are first *selected* and then *arranged.*

(12) A society consists of 50 members, 10 of whom are physicians. In how many ways can a committee of 6 members be selected so as to include *at least* one physician?

Six members can be selected from the whole society in

$$\frac{\underline{|50}}{\underline{|6}\,\underline{|44}} \text{ ways.}$$

Six members can be selected from the whole society, so as to include *no physician*, by choosing them all from the 40 members who are not physicians, and this can be done in

$$\frac{\underline{|40}}{\underline{|6}\,\underline{|34}} \text{ ways.}$$

Hence, $\dfrac{\underline{|50}}{\underline{|6}\,\underline{|44}} - \dfrac{\underline{|40}}{\underline{|6}\,\underline{|34}}$ is the number of ways of selecting the committee so as to include at least one physician.

306. Greatest number of Selections. To find for what value of r the number of selections of n things, taken r at a time, is the greatest.

The formula

$$C_{n,r} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{1 \times 2 \times 3 \times \cdots\cdots r}$$

may be written

$$C_{n,r} = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \cdots\cdots \frac{n-r+1}{r}.$$

The numerators of the factors on the right side of this equation begin with n, and form a descending series with the common difference 1; and the denominators begin with 1, and form an ascending series with the common difference 1. Therefore, from some point in the series, these factors

become less than 1. Hence, the maximum product is reached when that product includes *all* the factors *greater* than 1.

I. When n is an *odd* number, the numerator and the denominator of each factor will be alternately both odd and both even; so that the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 2. Hence, in this case, r must have such a value that

$$n-r+1=r+2, \text{ or } r=\frac{n-1}{2}.$$

II. When n is an *even* number, the numerator of the first factor will be even and the denominator odd; the numerator of the second factor will be odd and the denominator even; and so on, alternately; so that the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 1. Hence, in this case, r must have such a value that

$$n-r+1=r+1, \text{ or } r=\frac{n}{2}.$$

(1) What value of r will give the greatest number of selections out of 7 things?

Here n is odd, and $r=\frac{n-1}{2}=\frac{7-1}{2}=3.$

$$\therefore\ s=\frac{7\times6\times5}{1\times2\times3}=35. \quad \textit{Ans.}$$

If $r=4$, then $$s=\frac{7\times6\times5\times4}{1\times2\times3\times4}=35.$$

When the number of things is *odd*, there will be two equal numbers of selections; namely, when the number of things taken together is *just under* and *just over one-half* of the whole number of things.

(2) What value of r will give the greatest number of selections out of 8 things?

Here n is even, and $r = \frac{n}{2} = \frac{8}{2} = 4.$

$$\therefore s = \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70. \quad \textit{Ans.}$$

So that, when the number of things is *even*, the number of selections will be greatest when *one-half* of the whole are taken together.

307. Division into Groups. The number of different ways in which $p + q$ things all different can be divided into *two* groups of p things and q things, respectively, is the same as the number of ways in which p things can be *selected* from $p + q$ things, or $\dfrac{\lfloor p+q}{\lfloor p \lfloor q}$.

For, to each selection of p things *taken* corresponds a selection of q things *left*, and each selection therefore effects the division into the required groups.

(1) In how many ways can 18 men be divided into 2 groups of 6 and 12 each?

$$\frac{\lfloor 18}{\lfloor 6 \lfloor 12}$$

(2) A boat's crew consists of 8 men, of whom 2 can row only on the stroke side of the boat, and 3 can row only on the bow side. In how many ways can the crew be arranged?

There are left 3 men who can row on either side; 2 of these must row on the stroke side, and 1 on the bow side.

The number of ways in which these three can be divided is

$$\frac{\lfloor 3}{\lfloor 2 \lfloor 1} = 3 \text{ ways.}$$

When the stroke side is completed, the 4 men can be arranged in $\lfloor 4$ ways; likewise, the 4 men of the bow side can be arranged in $\lfloor 4$ ways. Hence the arrangement can be made in

$$3 \times \lfloor 4 \times \lfloor 4 = 1728 \text{ ways.}$$

308. The number of different ways in which $p+q+r$ things all different can be divided into *three* groups of p things, q things, and r things, respectively, is $\dfrac{\lfloor p+q+r}{\lfloor p \lfloor q \lfloor r}$.

For, $p+q+r$ things may be divided into two groups of p things and $q+r$ things in $\dfrac{\lfloor p+q+r}{\lfloor p \lfloor q+r}$ ways; then, the group of $q+r$ things may be divided into two groups of q things and r things in $\dfrac{\lfloor q+r}{\lfloor q \lfloor r}$ ways; hence the division into three groups may be effected in

$$\frac{\lfloor p+q+r}{\lfloor p \lfloor q+r} \times \frac{\lfloor q+r}{\lfloor q \lfloor r} \text{ or } \frac{\lfloor p+q+r}{\lfloor p \lfloor q \lfloor r} \text{ ways.}$$

And so on for any number of groups.

Ex. In how many ways can a company of 100 soldiers be divided into three squads of 50, 30, and 20, respectively?

The answer is $\dfrac{\lfloor 100}{\lfloor 50 \lfloor 30 \lfloor 20}$ ways.

309. When the number of things is the *same* in two or more groups, *and there is no distinction to be made between these groups*, the number of ways given by the preceding section is too large.

Ex. Divide the letters a, b, c, d, into two groups of two letters each.

The number of ways given by § 307 is $\dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} = 6$; these ways are:

I.	*ab*	*cd.*	IV.	*bc*	*ad.*
II.	*ac*	*bd.*	V.	*bd*	*ac.*
III.	*ad*	*bc.*	VI.	*cd*	*ab.*

Since there is no distinction between the groups, IV. is the same as III., V. the same as II., and VI. the same as I. Hence, the correct answer is $\frac{1}{2} \times \frac{|\underline{4}}{|\underline{2}\,|\underline{2}}$ or 3.

If, however, a distinction is to be made between the two groups in any one division, the answer is 6.

In the case of three similar groups the result given by § 308 is to be divided by $|\underline{3}$, the number of ways in which three groups can be arranged among themselves; in the case of four groups by $|\underline{4}$; and so on for any number of groups.

(1) In how many ways can 18 men be divided into 2 groups of 9 each?

According to § 307, the answer would be $\frac{|\underline{18}}{|\underline{9}\,|\underline{9}}$.

The two groups, considered as groups, have no distinction; therefore, permuting them gives no new arrangement, and the true result is obtained by dividing the preceding by $|\underline{2}$, and is $\frac{|\underline{18}}{|\underline{2}\,|\underline{9}\,|\underline{9}}$.

If any condition be added that will make the two groups *different*, if, for example, one group wear red badges and the other blue, then the answer will be $\frac{|\underline{18}}{|\underline{9}\,|\underline{9}}$.

(2) In how many ways can a pack of 52 cards be divided equally among four players, A, B, C, D?

Here the assignment of a particular group to a *different* player makes the *division* different, and there is therefore a distinction between the groups; the answer is

$$\frac{|\underline{52}}{|\underline{13}\,|\underline{13}\,|\underline{13}\,|\underline{13}}.$$

(3) In how many ways can 52 cards be divided into 4 piles of 13 each?

Here there is no distinction between the groups, and the answer is

$$\frac{\lfloor 52}{\lfloor 4 \lfloor 13 \lfloor 13 \lfloor 13 \lfloor 13}.$$

Exercise 47.

1. How many numbers of five figures each can be formed with the digits 1, 2, 3, 4, 5, no digit being repeated?

2. How many *even* numbers of four figures each can be formed with the digits 1, 2, 3, 4, 5, 6, no digit being repeated?

3. How many *odd* numbers between 1000 and 5000 can be formed with the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, no figure being repeated? How many of these numbers will be divisible by 5?

4. How many three-lettered words can be made from the alphabet, no letter being repeated in the same word?

5. In how many ways can 4 persons, A, B, C, D, sit at a round table?

6. In how many ways can 6 persons form a ring?

7. How many words can be made with 9 letters, 3 letters remaining inseparable and keeping the same order?

8. What will be the answer to the preceding problem if the 3 inseparable letters can be arranged in any order?

9. A captain, having under his command 60 men, wishes to form a guard of 8 men. In how many different ways can the guard be formed?

10. A detachment of 30 men must furnish each night a guard of 4 men. For how many nights can a different guard be formed, and how many times will each soldier serve?

11. Out of 12 Democrats and 16 Republicans, how many different committees can be formed, each committee consisting of 3 Democrats and 4 Republicans?

12. Out of 26 Republicans and 14 Democrats, how many different committees can be formed, each committee consisting of 10 Republicans and 8 Democrats?

13. There are m different things of one kind and n different things of another kind; how many different sets can be made, each set containing r things of the first kind and s of the second?

14. With 12 consonants and 6 vowels, how many different words can be formed consisting of 3 different consonants and 2 different vowels, any arrangement of letters being considered a word?

15. With 10 consonants and 6 vowels, how many words can be formed, each word containing 5 consonants and 4 vowels?

16. How many words can be formed with 20 consonants and 6 vowels, each word containing 3 consonants and 2 vowels, the vowels occupying the second and fourth places?

17. An assembly of stockholders, composed of 40 merchants, 20 lawyers, and 10 physicians, wishes to elect a commission of 4 merchants, 1 physician, and 2 lawyers. In how many ways can the commission be formed?

18. Of 8 men forming a boat's crew, one is selected as stroke. How many arrangements of the rest are possible? When the 4 men who row on each side are decided on, how many arrangements are still possible?

19. A boat's crew consists of 8 men. Either A or B must row stroke. Either B or C must row bow. D can pull only on the starboard side. In how many ways can the crew be seated?

NOTE. Stroke and bow are on opposite sides of the boat.

20. A boat's crew consists of 8 men. Of these, 3 can row only on the port side, and 2 only on the starboard side. In how many ways can the crew be seated?

21. Of a base ball nine, either A or B must pitch; either B or C must catch; D, E, and F play in the field. In how many ways can the nine be arranged?

22. How many signals may be made with 8 flags of different colors, which can be hoisted either singly, or any number at a time one above another?

23. Of 30 things, how many must be taken together, in order that having that number for selection, there may be the greatest possible variety of choice?

24. The number of combinations of $n+2$ objects, taken 4 at a time, is to the number of combinations of n objects, taken 2 at a time, as 11 is to 1. Find n.

25. The number of combinations of n things, taken r together, is 3 times the number taken $r-1$ together, and half the number taken $r+1$ together. Find n and r.

26. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack. How many cards are there in the pack?

27. It is proposed to divide 15 objects into lots, each lot containing 3 objects. In how many ways can the lots be made?

In the preceding sections we have considered only problems in which the things were all different. We proceed to problems in which some of the things are alike.

310.* Arrangements, Repetitions allowed. Suppose we have n letters, which are all different, and that *repetitions* are allowed.

Then, in making any arrangement, the first place can be filled in n ways.

When the first place has been filled, the second place can be filled in n ways, since repetitions are allowed. Hence the first two places can be filled in $n \times n$, or n^2, ways (§ 297).

Similarly, the first three places can be filled in $n \times n \times n$, or n^3, ways (§ 298).

In general, r places can be filled in n^r ways; or, *the number of arrangements of* n *different things taken* r *at a time, when repetitions are allowed, is* n^r.

(1) How many three-lettered words can be made from the alphabet, when repetitions are allowed?

Here the first place can be filled in 26 ways; the second place in 26 ways; and the third place in 26 ways. The number of words is, therefore, $26^3 = 17,576$. *Ans.*

(2) In the common system of notation, how many numbers can be formed, each number consisting of not more than 5 figures?

Each of the possible numbers may be regarded as consisting of 5 figures, by prefixing zeros to the numbers consisting of less than 5 figures. Thus, 247 may be written 00247.

Hence, every possible arrangement of 5 figures out of the 10 figures, except 00000, will give one of the required numbers; and the answer is $10^5 - 1 = 99,999$; that is, all the numbers between 0 and 100,000.

311.* Arrangements, Things alike, All together. Consider the number of arrangements of the letters a, a, b, b, b, c, d.

Suppose the a's to be different and the b's to be different, and distinguish between them by a_1, a_2, b_1, b_2, b_3.

The seven letters can now be arranged in $\lfloor 7$ ways (§ 300).

Now suppose the two a's to become alike, and the three b's to become alike. Then, where we before had $\lfloor 2$ arrangements of the a's among themselves, we now have but one arrangement, aa; and where we before had $\lfloor 3$ arrangements of the b's among themselves, we now have but one arrangement, bbb.

Hence, the number of arrangements is $\dfrac{\lfloor 7}{\lfloor 2 \lfloor 3} = 420.$

In general the number of arrangements of n *things, of which* p *are alike,* q *others are alike, and* r *others are alike, ..., is*

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r \ldots\ldots}.$$

(1) In how many ways can the letters of the word *College* be arranged?

If the two l's were different and the two e's were different, the number of ways would be $\lfloor 7$. Instead of two arrangements of the two l's, we have but one arrangement, ll; and instead of two arrangements of the two e's, we have but one arrangement, ee. Hence, the number of ways is $\dfrac{\lfloor 7}{\lfloor 2 \lfloor 2} = 1260.$ *Ans.*

(2) In how many ways can the letters of the word *Mississippi* be arranged?

$$\frac{\lfloor 11}{\lfloor 4 \lfloor 4 \lfloor 2} = 34{,}560. \quad \textit{Ans.}$$

(3) In how many different orders can a row of 4 white balls and 3 black balls be arranged?

$$\frac{\lfloor 7}{\lfloor 4 \lfloor 3} = 35. \quad \textit{Ans.}$$

312.* Selections, Repetitions allowed. We shall illustrate by two examples the method of solving problems which come under this head.

(1) In how many ways can a selection of 3 letters be made from the letters a, b, c, d, e, if repetitions are allowed?

The selections will be of three classes:

(*a*) All three letters alike.
(*b*) Two letters alike.
(*c*) The three letters all different.

(*a*) There will be 5 selections, since any one of the five letters may be taken three times.

(*b*) Any one of the five letters may be taken twice, and with these may be put any one of the other four letters. Hence, the number of selections is 5×4, or 20.

(*c*) The number of selections (§ 302) is $\frac{5 \times 4 \times 3}{1 \times 2 \times 3}$, or 10. Hence, the total number of selections is $5 + 20 + 10 = 35$. *Ans.*

(2) How many different throws can be made with 4 dice?

The throws may be divided into five classes:

(*a*) All four dice alike.
(*b*) Three dice alike.
(*c*) Two dice alike, and the other two alike.
(*d*) Two dice alike, and the other two different.
(*e*) The four dice different.

(*a*) There are six throws.

(*b*) Any of the six numbers may be taken three times, and with these may be put any other of the five remaining numbers. Hence, the number of throws is 6×5, or 30.

(*c*) Any two of the six pairs of doublets may be selected. Hence, the number of throws is $\frac{6 \times 5}{1 \times 2}$, or 15.

(*d*) Any pair of doublets may be put with any selection of two different numbers from the remaining five. Hence, the number of throws is $6 \times \frac{5 \times 4}{1 \times 2} = 60$.

(*e*) The number of throws is $\frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 15$.

The answer is, then, $6 + 30 + 15 + 60 + 15 = 126$.

313.* Selections and Arrangements, Things alike. We shall illustrate by an example the method of solving problems which come under this head.

How many selections of four letters each can be made from the letters in *Proportion?* How many arrangements of four letters each?

There are 10 letters as follows:

$$\begin{array}{cccccc} o & p & r & t & i & n \\ o & p & r & & & \\ o & & & & & \end{array}$$

Selections:

The selections may be divided into four classes:

(*a*) Three letters alike.
(*b*) Two letters alike, two others alike.
(*c*) Two letters alike, other two different.
(*d*) Four letters different.

(*a*) With the three *o*'s we may put any one of the five other letters, giving 5 selections.

(*b*) We may choose any two out of the three pairs, *o, o*; *p, p*; *r, r*.

$$\frac{3 \times 2}{1 \times 2} = 3 \text{ selections.}$$

(*c*) With any one of the three pairs we can put any two of the five remaining letters in the first line.

$$3 \times \frac{5 \times 4}{1 \times 2} = 30 \text{ selections.}$$

(*d*) $$\frac{6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4} = 15 \text{ selections.}$$

Hence, the total number of *selections* is

$$5 + 3 + 30 + 15 = 53.$$

Arrangements:

(*a*) Each selection can be arranged in $\frac{\lfloor 4}{\lfloor 3} = 4$ ways.

$$5 \times 4 = 20 \text{ arrangements.}$$

(*b*) Each selection can be arranged in $\frac{\lfloor 4}{\lfloor 2 \lfloor 2} = 6$ ways.

$3 \times 6 = 18$ arrangements.

(*c*) Each selection can be arranged in $\frac{\lfloor 4}{\lfloor 2} = 12$ ways.

$30 \times 12 = 360$ arrangements.

(*d*) Each selection can be arranged in $\lfloor 4 = 24$ ways.

$15 \times 24 = 360$ arrangements.

Hence, the total number of *arrangements* is

$$20 + 18 + 360 + 360 = 758.$$

314.* Total Number of Selections.

I. *The whole number of ways in which a selection (of some, or all) can be made from* n *different things is* $2^n - 1$.

For each thing can be either taken or left; that is, can be disposed of in two ways.

There are n things; hence (§ 298) they can all be disposed of in 2^n ways. But among these ways is included the case in which all are rejected; and this case is inadmissible.

Hence, the number of ways of making a selection is $2^n - 1$.

(1) In a shop window 20 different articles are exposed for sale. What choice has a purchaser?

$2^{20} - 1 = 1,048,575$. *Ans.*

(2) How many different amounts can be weighed with 1 lb., 2 lb., 4 lb., 8 lb., and 16 lb. weights?

$2^5 - 1 = 31$. *Ans.*

(Let the student write out the 31 weights.)

II. *The whole number of ways in which a selection can be made from* $p + q + r \cdots\cdots$ *things, of which* p *are alike,* q *are alike,* r *are alike, etc., is* $(p + 1)(q + 1)(r + 1 \cdots\cdots - 1)$.

For the set of p things may be disposed of in $p+1$ ways, since none of them may be taken, or 1, 2, 3,....., or p, may be taken.

In like manner, the q things may be disposed of in $q+1$ ways; the r things in $r+1$ ways; and so on.

Hence (§ 298) all the things may be disposed of in $(p+1)(q+1)(r+1)\cdots\cdots$ ways.

But the case in which *all* the things are rejected is inadmissible; hence, the whole number of ways is

$$(p+1)(q+1)(r+1)\cdots\cdots-1.$$

Ex. In how many ways can 2 boys divide between them 10 oranges all alike, 15 apples all alike, and 20 peaches all alike?

Here, the case in which the first boy takes none, and the case in which the second boy takes none, must be rejected.

Therefore, the answer is one less than the result, according to II.

$$11\times 16\times 21-2=3694. \quad \textit{Ans.}$$

Exercise 48.*

1. How many three-lettered words can be made from the 6 vowels when repetitions are allowed?

2. A railway signal has 3 arms, and each arm may take 4 different positions, including the position of rest. How many signals in all can be made?

3. In how many different orders can a row of 7 white balls, 2 red balls, and 3 black balls be arranged?

4. In how many ways can the letters of the word *Mathematics*, taken all together, be arranged?

5. How many different signals can be made with 10 flags, of which 3 are white, 2 red, and the rest blue, always hoisted all together and one above another?

6. How many signals can be made with 7 flags, of which 2 are red, 1 white, 3 blue, and 1 yellow, always displayed all together and one above another?

7. In how many ways can 5 letters be selected from a, b, c, d, e, f, if each letter may be taken once, twice, up to five times, in making the selection?

8. In how many ways can 6 rugs be selected at a shop where 2 kinds of rugs are sold?

9. How many dominos are there in a set numbered from double blank to double ten?

10. In how many ways can 3 letters be selected from n different letters, when repetitions are allowed?

11. Five flags of different colors can be hoisted either singly, or any number at a time, one above another. How many different signals can be made with them?

12. If there are m kinds of things, and 1 thing of the first kind, 2 of the second, 3 of the third, and so on, in how many ways can a selection be made?

13. How many selections of 6 letters each can be made from the letters in *Democracy?* How many arrangements of 6 letters each?

14. If of $p+q+r$ things, p are alike, and q are alike, and the rest different, show that the total number of selections is $(p+1)(q+1)2^r-1$.

15. Show that the total number of arrangements of $2n$ letters, of which some are a's and the rest b's, is greatest when the number of a's is equal to the number of b's.

16. If in a given number the prime factor a occurs m times, the prime factor b, n times, the prime factor c, p times, find the number of different divisors of the given number.

CHAPTER XXII.

CHANCE.

315. Definitions. If an event can happen in a ways and fail in b ways, and all these $a+b$ ways are *equally likely* to occur; if, also, one, *and only one*, of these $a+b$ ways *can* occur, and one *must* occur; then, the **chance** of the event *happening* is $\frac{a}{a+b}$, and the chance of the event *failing* is $\frac{b}{a+b}$.

Thus, let the event be the throwing of an even number with a single die.

The event can happen in 3 ways, by the die turning up a two, a four, or a six; and fail in 3 ways, by the die turning up a one, a three, or a five; and all these 6 ways are equally likely to occur.

Moreover, one, and only one, of these 6 ways *can* occur, and one *must* occur (for it is assumed that the die is to be thrown).

Consequently, by the definition, the chance of throwing an even number is $\frac{3}{3+3}$, or $\frac{1}{2}$; and the chance of throwing a number not even, that is odd, is $\frac{3}{3+3}$, or $\frac{1}{2}$.

The above may be regarded as giving a definition of the term *chance* as that term is used in mathematical works. Instead of *chance*, *probability* is often used.

316. Odds. In the case of the event in § 315 the *odds* are said to be a to b *in favor of* the event, if a is greater than b; and b to a *against* the event, if b is greater than a.

If $a=b$, the odds are said to be *even* on the event.

Thus the odds are 5 to 1 against throwing a six in one throw with a single die, since there are 5 unfavorable ways and 1 favorable way, and all these 6 ways are equally likely to occur.

317. Rules. From the definitions it is evident that:

The chance of an event happening is expressed by the fraction of which the numerator is the number of favorable ways, and the denominator the whole number of ways favorable and unfavorable.

For example, take the throwing of a six with a single die. The number of favorable ways is 1; the whole number of ways is 6. Hence, the chance of throwing a six is $\frac{1}{6}$.

The chance of an event not happening is expressed by the fraction of which the numerator is the number of unfavorable ways, and the denominator the whole number of ways favorable and unfavorable.

For example, take the throwing of a six with a single die. The number of unfavorable ways is 5; the whole number of ways is 6. Hence, the chance of not throwing a six is $\frac{5}{6}$.

318. Certainty. If the event is *certain* to happen, there are no ways of failing, and $b = 0$. The chance of the event happening is then $\dfrac{a}{a+0} = 1$. Hence *certainty* is expressed by 1.

It is to be observed that the fraction which expresses a *chance* (or *probability*) is less than 1, unless the event is certain to happen, in which case the chance of the event happening is 1.

319. Since
$$\frac{a}{a+b} + \frac{b}{a+b} = 1,$$
we have
$$\frac{b}{a+b} = 1 - \frac{a}{a+b}.$$

Hence, if p is the chance of an event happening, $1 - p$ is the chance of the event failing.

320. Examples. Simple Event.

(1) What is the chance of throwing double sixes in one throw with two dice?

Each die may fall in 6 ways, and all these ways are equally likely to occur. Hence, the two dice may fall in 6×6, or 36, ways (§ 297), and these 36 ways are all equally likely to occur. Moreover, only *one* of the 36 ways *can* occur, and one *must* occur.

There is only one way which will give double sixes. Hence the chance of throwing double sixes is $\frac{1}{36}$.

REMARK. It may seem as though the number of ways in which the dice can fall ought to be 21, the number of different throws that can be made with two dice. These throws, however, are not all *equally likely* to occur.

To obtain ways that are equally likely to occur we must go back to the case of a single die. One die can fall in 6 ways, and from the *construction of the die* it is evident that these 6 ways are all equally likely to occur.

Also the second die can fall in 6 ways, all equally likely to occur. Hence, the two dice can fall in 36 ways, all equally likely to occur (§ 297).

In this case the throw, first die five second die six, is considered a different throw from first die six second die five. Consequently, the chance of throwing a five and a six is $\frac{2}{36}$, or $\frac{1}{18}$, while the chance of throwing double sixes is only $\frac{1}{36}$. This verifies the statement already made, that the 21 different throws are not all equally likely to occur.

(2) What is the chance of throwing one, and only one, five in one throw with two dice?

The whole number of ways, all equally likely to occur, in which the dice can fall is 36. In 5 of these 36 ways the first die will be a five, and the second die not a five; in 5 of these 36 ways the second die will be a five, and the first not a five. Hence, in 10 of these ways one die, and only one die, will be a five; and the required chance is $\frac{10}{36}$, or $\frac{5}{18}$.

The odds are 13 to 5 against the event.

(3) In the same problem, what is the chance of throwing *at least* one five?

Here we have to include also the way in which both dice fall fives, and the required chance is $\frac{11}{36}$.

The odds are 25 to 11 against the event.

(4) What is the chance of throwing a total of 5 in one throw with two dice?

The whole number of ways, all equally likely to occur, in which the dice can fall is 36. Of these ways 4 give a total of 5; viz., 1 and 4, 2 and 3, 3 and 2, 4 and 1. Hence, the required chance is $\frac{4}{36}$, or $\frac{1}{9}$.

The odds are 8 to 1 against the event.

(5) From an urn containing 5 black and 4 white balls, 3 balls are to be drawn at random. Find the chance that 2 balls will be black and 1 white.

There are 9 balls in the urn. The whole number of ways in which 3 balls can be selected from 9 is $\frac{9 \times 8 \times 7}{1 \times 2 \times 3}$, or 84.

From the 5 black balls 2 can be selected in $\frac{5 \times 4}{1 \times 2}$, or 10, ways; from the 4 white balls 1 can be selected in 4 ways; hence, 2 black balls and 1 white ball can be selected in 10×4, or 40, ways.

The required chance is $\frac{40}{84} = \frac{10}{21}$.

The odds are 11 to 10 against the event.

(6) From a bag containing 10 balls, 4 are drawn and replaced; then 6 are drawn. Find the chance that the 4 first drawn are among the 6 last drawn.

The second drawing could be made altogether in

$$\frac{\lfloor 10}{\lfloor 6 \lfloor 4} = 210 \text{ ways.}$$

But the drawing can be made so as to include the 4 first drawn in

$$\frac{\lfloor 6}{\lfloor 2 \lfloor 4} = 15 \text{ ways,}$$

since the only choice consists in selecting 2 balls from the 6 not previously drawn. Hence, the required chance is $\frac{15}{210} = \frac{1}{14}$.

(7) If 4 coppers are tossed, what is the chance that exactly 2 will turn up heads?

Since each coin may fall in 2 ways, the 4 coins may fall in $2^4 = 16$ ways (§ 298). The 2 coins to turn up heads can be selected from the 4 coins in $\frac{4 \times 3}{1 \times 2} = 6$ ways. Hence, the required chance is $\frac{6}{16} = \frac{3}{8}$.

The odds are 5 to 3 against the event.

(8) In one throw with two dice which sum is more likely to be thrown, 9 or 12?

Out of the 36 possible ways of falling, *four* give the sum 9 (namely, $6+3$, $3+6$, $5+4$, $4+5$), and *only one* way gives 12 (namely, $6+6$). Hence, the chance of throwing 9 is *four times* that of throwing 12.

NOTE. It will be observed in the above examples that we sometimes use arrangements and sometimes use selections. In some problems the former, in some problems the latter, will give the ways which are all *equally likely* to occur.

In some problems we can use either selections or arrangements.

Exercise 49.

1. The chance of an event happening is $\frac{4}{7}$. What are the odds in favor of the event?

2. If the odds are 10 to 1 against an event, what is the chance of its happening?

3. The odds against an event are 3 to 1. What is the chance of the event happening?

4. The chance of an event happening is $\frac{2}{9}$. Find the odds against the event.

5. In one throw with a pair of dice what number is most likely to be thrown? Find the odds against throwing that number.

6. Find the chance of throwing doublets in one throw with a pair of dice.

7. If 4 cards are drawn from a pack of 52 cards, what is the chance that there will be one of each suit?

8. If 4 cards are drawn from a pack of 52 cards, what is the chance that they will all be hearts?

9. If 10 persons stand in a line, what is the chance that 2 assigned persons will stand together?

10. If 10 persons form a ring, what is the chance that 2 assigned persons will stand together?

11. Three balls are to be drawn from an urn containing 5 black, 3 red, and 2 white balls. What is the chance of drawing 1 red and 2 black balls?

12. In a bag are 5 white and 4 black balls. If 4 balls are drawn out, what is the chance that they will be all of the same color?

13. If 2 tickets are drawn from a package of 20 tickets marked 1, 2, 3, ..., what is the chance that both will be marked with *odd* numbers?

14. A bag contains 3 white, 4 black, and 5 red balls; 3 balls are drawn. Find the odds against the 3 being of three different colors.

15. Show that the odds are 35 to 1 against throwing 16 in a single throw with 3 dice.

16. There are 10 tickets numbered 1, 2, ... 9, 0. Three tickets are drawn at random. Find the chance of drawing a total of 22.

17. Find the probability of throwing 15 in one throw with 3 dice.

18. With 3 dice, what are the relative chances of throwing a doublet and a triplet?

19. If 3 cards are drawn from a pack of 52 cards, what is the chance that they will be king, queen, and knave?

321. Dependent and Independent Events. Thus far we have considered only single events. We proceed to cases in which there are two or more events.

Two or more events are **dependent** or **independent**, according as the happening (or failing) of one event *does* or *does not* affect the happening (or failing) of the other events.

Thus, throwing a six and throwing a five in any particular throw with one die are *dependent* events, since the happening of one *excludes* the happening of the other.

But, with two dice, throwing a six with one die and throwing a five with the other are *independent* events, since the happening of one has no effect upon the happening of the other.

322. Events Mutually Exclusive. When several dependent events are so related that one, and *only one,* of the events can happen, the events are said to be mutually exclusive.

Thus, let a single die be thrown, and regard its falling one up, two up, three up, etc., as six different events. Then, these six events are evidently mutually exclusive.

323. *If there are several events of which one, and only one, can happen, the chance that one will happen is the* **sum** *of the respective chances of happening.*

To prove this, let a, a', a'', ... be the number of ways favorable to the first, second, third, ... events, respectively, and m the number of ways unfavorable to *all* the events, these $a + a' + a'' + \cdots + m$ ways being all equally likely to occur, and such that one *must* occur.

Represent by n the sum $a + a' + a'' + \cdots + m$.

Of the n ways which may occur, a, a', a'', ... ways are favorable to the first, second, third, ... events, respectively. Hence, the respective chances of happening are

$$\frac{a}{n}, \frac{a'}{n}, \frac{a''}{n}, \ldots$$

Of the n ways which may occur, $a + a' + a'' + \cdots$ ways are favorable to the happening of *some one* of the events. Hence, the chance that some one of the events will happen is

$$\frac{a + a' + a'' + \cdots}{n}, \text{ or } \frac{a}{n} + \frac{a'}{n} + \frac{a''}{n} + \cdots$$

If, then, p, p', p'', ... be the respective chances of happening of the first, second, third, ..., of several mutually exclusive events, the chance that *some one* of the events will happen is $p + p' + p'' + \cdots$

Thus, let the throwing of a two, a four, and a six, with a single die, be three events. These three events are evidently mutually exclusive.

There are 6 ways, all equally likely to occur, in which the die can fall; of these 6 ways one must occur and only one can occur.

The chance of throwing a two is $\frac{1}{6}$; of throwing a four, $\frac{1}{6}$; of throwing a six, $\frac{1}{6}$; since there is but one favorable way in each case.

The chance of throwing an even number is $\frac{3}{6}$, since 3 out of the 6 ways are favorable ways.

But $\frac{3}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$; hence $\frac{3}{6}$ is the sum of the respective chances of throwing a two. a four, a six. Cf. § 315, Ex.

324. Compound Events. If there are two or more events, the happening of them together, or in succession, may be regarded as a compound event.

Thus, the throwing of double sixes with a pair of dice may be regarded as a compound event compounded of the throwing of a six with the first die and the throwing of a six with the second die.

325. Concurring Independent Events. *The chance that two or more independent events will happen together is the* **product** *of the respective chances of happening.*

To prove this, let a and a' be the number of ways favorable to, and b and b' the number of ways unfavorable to, the first and second events respectively; the $a + b$ ways being all equally likely to occur, and such that one must

occur, and only one can occur; and the $a'+b'$ ways being all equally likely to occur, and such that one must occur, and only one can occur.

Then, the respective chances of happening are $\frac{a}{a+b}$ and $\frac{a'}{a'+b'}$; and the respective chances of failing are $\frac{b}{a+b}$ and $\frac{b'}{a'+b'}$. Represent the former by p and p'; then the latter will be $1-p$ and $1-p'$.

Consider the compound event. There are (§ 297) $(a+b)(a'+b')$ ways, all equally likely to occur, of which one *must* occur, and only one *can* occur.

The number of ways in which both events can happen is aa'; hence, the chance that both events will happen is

$$\frac{aa'}{(a+b)(a'+b')}=\frac{a}{a+b}\times\frac{a'}{a'+b'}=pp'.$$

Similarly, the chance that both events will fail is

$$\frac{bb'}{(a+b)(a'+b')}=(1-p)(1-p');$$

the chance that the first will happen and the second fail is

$$\frac{ab'}{(a+b)(a'+b')}=p(1-p');$$

the chance that the first will fail and the second happen is

$$\frac{ba'}{(a+b)(a'+b')}=(1-p)p'.$$

Similarly for three or more events.

326. Successive Dependent Events. By a slight change in the meaning of the symbols of § 325, we can find the chance of the happening together of two or more *dependent* events.

For, suppose that, *after the first event has happened,* the second event can follow in a' ways and not follow in b' ways. Then the two events can happen in $\dfrac{aa'}{(a+b)(a'+b')}$ ways; and so on as in § 325.

Hence, if p is the chance that the first event will happen, and p' the chance that after the first event has happened the second will follow, pp' is the chance of both happening; $(1-p)(1-p')$, the chance of both failing; and so on.

Similarly for three or more events.

327. Examples. (1) What is the chance of throwing double sixes in one throw with two dice?

Regard this as a *compound* event. The chance that the first die will turn up a six is $\frac{1}{6}$; the chance that the second die will turn up a six is $\frac{1}{6}$; the chance that both dice will turn up sixes is $\frac{1}{6}\times\frac{1}{6}$, or $\frac{1}{36}$.
The events are here *independent.* In Ex. 1, § 320, the throwing of double sixes is regarded as a *simple* event.

(2) What is the chance of throwing one, and only one, five, in a single throw with two dice?

The chance that the first die will be a five, and the second not a five, is $\frac{1}{6}\times\frac{5}{6}=\frac{5}{36}$; the chance that the first die will not be a five, and the second die a five, is $\frac{5}{6}\times\frac{1}{6}=\frac{5}{36}$. These two events are dependent and mutually exclusive, and the chance that one or the other of them will happen is (§ 323) $\frac{5}{36}+\frac{5}{36}=\frac{5}{18}$. Cf. Ex. 2, § 320.

(3) What is the chance of throwing a total of 5 in one throw with two dice?

There are 4 ways of throwing 5: 1 and 4, 2 and 3, 3 and 2, 4 and 1. The chance of each of these ways happening is $\frac{1}{36}$. The events are mutually exclusive; hence, the chance of some one happening is (§ 323) $\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{1}{9}$. Cf. Ex. 4, § 320.

(4) A bag contains 3 balls, 2 of which are white; another bag contains 6 balls, 5 of which are white. If a person is to draw one ball from each bag, what is the chance that both balls drawn will be white?

The chance that the ball drawn from the first bag will be white is $\frac{2}{3}$; the chance that the ball drawn from the second bag will be white is $\frac{5}{6}$. The events are independent; hence, the chance that both balls will be white is $\frac{2}{3} \times \frac{5}{6} = \frac{5}{9}$ (§ 325).

(5) In the last example, if all the balls are in one bag, and 2 balls are to be drawn, what is the chance that both balls will be white?

The chance that the first ball will be white is $\frac{7}{9}$; the chance that, after 1 white ball has been drawn, the second will be white, is $\frac{6}{8}$; the chance of drawing 2 white balls is (§ 326) $\frac{7}{9} \times \frac{6}{8} = \frac{7}{12}$.

(6) The chance that A can solve this problem is $\frac{2}{3}$; the chance that B can solve it is $\frac{5}{12}$. If both try, what is the chance (1) that both solve it; (2) that A solves it, and B fails; (3) that A fails, and B solves it; (4) that both fail?

A's chance of success is $\frac{2}{3}$, A's chance of failure is $\frac{1}{3}$.
B's chance of success is $\frac{5}{12}$, B's chance of failure is $\frac{7}{12}$.
Therefore, the chance of (1) is $\frac{2}{3} \times \frac{5}{12} = \frac{10}{36}$;
the chance of (2) is $\frac{2}{3} \times \frac{7}{12} = \frac{14}{36}$;
the chance of (3) is $\frac{1}{3} \times \frac{5}{12} = \frac{5}{36}$;
the chance of (4) is $\frac{1}{3} \times \frac{7}{12} = \frac{7}{36}$.
The sum of these four chances is $\frac{10}{36} + \frac{14}{36} + \frac{5}{36} + \frac{7}{36} = 1$, as it ought to be, since one of the four results is *certain* to happen.

(7) In Ex. (6) what is the chance that the problem will be solved?

The chance that *both fail* is $\frac{7}{36}$. Hence, the chance that *both do not fail*, or that the problem will be solved, is $1 - \frac{7}{36} = \frac{29}{36}$ (§ 319).

(8) From an urn containing 5 black and 4 white balls, 3 balls are to be drawn at random. Find the chance that of the 3 balls 2 will be black and 1 white.

There are 9 balls in the urn. Suppose the balls to be drawn 1 at a time. The white ball may be either the first, second, or third ball drawn. In other words, one white ball and two black balls may be drawn in $\frac{\lfloor 3}{\lfloor 1 \lfloor 2} = 3$ ways (§ 307).

The chance of the order, white black black, is $\frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{10}{63}$.

The chance of the order, black white black, is $\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{10}{63}$.

The chance of the order, black black white, is $\frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{10}{63}$.

Hence, the required chance is $\frac{10}{63} + \frac{10}{63} + \frac{10}{63} = \frac{10}{21}$ (§ 323).

The method of Ex. 5, § 320, is, however, recommended for problems of this nature.

(9) When 6 coins are tossed, what is the chance that *one*, and *only one*, will fall with the head up?

The chance that the first alone falls with the head up is (§ 325) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64}$; the chance that the second alone falls with the head up is $\frac{1}{64}$; and so on for each of the 6 coins.

Hence, the chance that some one coin, and only one coin, falls with the head up is $\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} = \frac{6}{64} = \frac{3}{32}$.

(10) When 6 coins are tossed, what is the chance that *at least* one will fall with the head up?

The chance that *all* will fall heads down is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{64}$. Hence, the chance that this will not happen is $1 - \frac{1}{64} = \frac{63}{64}$.

(11) A purse contains 9 silver dollars and 1 gold eagle, and another contains 10 silver dollars. If 9 coins are taken out of the first purse and put into the second, and then 9 coins are taken out of the second and put into the first purse, which purse now is the more likely to contain the gold coin?

The gold eagle will not be in the second purse unless it (1) was among the 9 coins taken out of the first and put into the second purse; (2) and *not* among the 9 coins taken out of the second and put into the first purse. The chance of (1) is $\frac{9}{10}$, and when (1) has happened, the chance of (2) is $\frac{10}{19}$. Hence, the chance of *both* happening is $\frac{9}{10} \times \frac{10}{19} = \frac{9}{19}$. Therefore, the chance that the eagle is in the second

purse is $\frac{9}{19}$, and the chance that it is in the first purse is $1 - \frac{9}{19} = \frac{10}{19}$. Since $\frac{10}{19}$ is greater than $\frac{9}{19}$, the gold coin is more likely to be in the first purse than in the second.

NOTE. The *expectation* from an uncertain event is the product of *the chance* that the event will happen by *the amount* to be realized in case the event happens.

(12) In a bag are 2 red and 3 white balls. A is to draw a ball, then B, and so on alternately; and whichever draws a white ball first is to receive \$10. Find their expectations.

A's chance of drawing a *white* ball at the first trial is $\frac{3}{5}$. B's chance of *having a trial* is equal to A's chance of drawing a *red* ball $= \frac{2}{5}$. In case A drew a red ball, there would be 1 red and 3 white balls left in the bag, and B's chance of drawing a white ball would be $\frac{3}{4}$. Hence, B's chance of having the trial and drawing a white ball is $\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$; and B's chance of drawing a red ball is $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$.

A's chance of *having a second trial* is equal to B's chance of drawing a *red* ball $= \frac{1}{10}$. In case B drew a red ball, there would be 3 white balls left, and A's chance of drawing a white ball would be *certainty*, or 1.

A's chance, therefore, is $\frac{3}{5} + \frac{1}{10} = \frac{7}{10}$; and B's chance is $\frac{3}{10}$.

A's expectation, then, is \$7, and B's \$3.

328. Repeated Trials. Given the chance of an event happening in one trial, to find the chance of its happening exactly once, twice, ... r times in n trials.

Let p be the chance of the event happening, and q the chance of the event failing, in one trial; so that $q = 1 - p$.

In n trials the event may happen exactly n times, $n - 1$ times, $n - 2$ times, ... down to no times. The respective chances of happening are as follows:

n times. The required chance by § 325 is p^n.

n − 1 times. The one failure may occur in any one of the n trials; that is, in n ways. The chance of any particular way occurring is $p^{n-1}q$; the required chance is therefore $np^{n-1}q$.

$n-2$ *times.* The two failures may occur in any two of the n trials; that is, in $\frac{n(n-1)}{\lfloor 2}$ ways. The chance of any particular way occurring is $p^{n-2}q^2$; the required chance is therefore $\frac{n(n-1)}{\lfloor 2}p^{n-2}q^2$.

r *times.* The $n-r$ failures may occur in any $n-r$ of the n trials; that is, in $\frac{\lfloor n}{\lfloor n-r \lfloor r}$ ways. The chance of any particular way occurring is $p^r q^{n-r}$; the required chance is therefore $\frac{\lfloor n}{\lfloor n-r \lfloor r}p^r q^{n-r}$.

Similarly, the chance of exactly r failures is $\frac{\lfloor n}{\lfloor r \lfloor n-r}p^{n-r}q^r$. The coefficients for r successes and r failures are the same by § 304.

If, then, $(p+q)^n$ be expanded by the binomial theorem, it is evident that the successive terms are the chances that the event will happen exactly n times, $n-1$ times, ... down to no times.

The chances that the event will happen *at least* r times in n trials is evidently $p^n + np^{n-1}q + \cdots\cdots \frac{\lfloor n}{\lfloor n-r \lfloor r}p^r q^{n-r}$.

NOTE. Since $p+q=1$, we have, whatever the value of n,

$$1 = p^n + np^{n-1}q + \cdots\cdots + npq^{n-1} + q^n,$$

a somewhat remarkable equation inasmuch as there exists but one relation betwen p and q, viz., $p+q=1$.

329. Examples.

(1) What is the chance of throwing a six exactly 3 times in 5 trials with a single die? At least 3 times?

There are to be two failures. The two failures may occur in any 2 of the 5 trials; that is, in $\frac{5\times 4}{2}$, or 10, ways. In any particular

way there will be 3 sixes and 2 failures, and the chance of this way occurring is $(\frac{1}{6})^3(\frac{5}{6})^2$; the chance of throwing *exactly* 3 sixes is therefore

$$10\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)=\frac{250}{6^5}=\frac{125}{3888}.$$

The chance of throwing *at least* 3 sixes is found by adding together the respective chances of throwing 5 sixes, 4 sixes, 3 sixes; and is $(\frac{1}{6})^5+5(\frac{1}{6})^4(\frac{5}{6})+10(\frac{1}{6})^3(\frac{5}{6})^2=\frac{23}{648}$.

(2) A's skill at a game, which cannot be drawn, is to B's skill as 3 to 4. If they play 3 games, what is the chance that A will win more games than B?

Their respective chances of winning a particular game are $\frac{3}{7}$ and $\frac{4}{7}$. For A to win more games than B, he must win all 3 games or 2 games. The chance that A wins all 3 games is $(\frac{3}{7})^3=\frac{27}{343}$. The chance that A wins any particular set of 2 games out of the 3 games, and that B wins the third game, is $(\frac{3}{7})^2\times(\frac{4}{7})$. As there are 3 ways of selecting a set of 2 games out of 3, the chance that A wins 2 games, and B the third game, is $3\times(\frac{3}{7})^2\times\frac{4}{7}=\frac{108}{343}$. Hence, the chance that A wins more than B is $\frac{27}{343}+\frac{108}{343}=\frac{135}{343}$.

(3) In the last example, find B's chance of winning more games than A.

B's chance of winning all 3 games is $(\frac{4}{7})^3=\frac{64}{343}$. The chance that B wins 2 games, and A the third game, is $3\times(\frac{4}{7})^2\times\frac{3}{7}=\frac{144}{343}$. Hence, B's chance of winning more games than A is $\frac{64}{343}+\frac{144}{343}=\frac{208}{343}$.

Notice that A's chance added to B's chance, $\frac{135}{343}+\frac{208}{343}$, is 1. Why should this be so?

(4) A and B throw with a single die alternately, A throwing first; and the one who throws an ace first is to receive a prize of \$110. What are their respective expectations?

The chance of winning the prize at the first throw is $\frac{1}{6}$; of winning at the second throw $\frac{5}{6}\times\frac{1}{6}$; of winning at the third throw $(\frac{5}{6})^2\times\frac{1}{6}$; of winning at the fourth throw $(\frac{5}{6})^3\times\frac{1}{6}$; and so on.

Hence, A's chance is $\frac{1}{6}+(\frac{5}{6})^2\frac{1}{6}+(\frac{5}{6})^4\frac{1}{6}+\cdots$, and B's chance is $(\frac{5}{6})\frac{1}{6}+(\frac{5}{6})^3\frac{1}{6}+(\frac{5}{6})^5\frac{1}{6}+\cdots$ Evidently B's chance is $\frac{5}{6}$ of A's chance.

Since A's chance + B's chance = 1, A's chance must be $\frac{6}{11}$ and B's $\frac{5}{11}$. A's expectation is $\frac{6}{11}$ of \$110, or \$66; and B's $\frac{5}{11}$ of \$110, or \$55.

This problem may also be solved as follows: A's chance, by § 131, is $\frac{6}{11}$ and B's $\frac{5}{11}$. Then A's expectation is \$66, and B's \$55.

Exercise 50.

1. One of two events must happen. If the chance of one is $\frac{2}{3}$ that of the other, find the odds on the first.

2. There are three events, A, B, C, one of which must happen. The odds are 3 to 8 on A, and 2 to 5 on B. Find the odds on C.

3. In one bag are 9 balls and in another 6; and in each bag the balls are marked 1, 2, 3, etc. What is the chance that on drawing one ball from each bag the two balls will have the same number?

4. What is the chance of throwing at least one ace in 2 throws with one die?

5. Find the probability of throwing a number greater than 9 in a single throw with a pair of dice.

6. The chance that A can solve a certain problem is $\frac{1}{4}$, and the chance that B can solve it is $\frac{2}{3}$. What is the chance that the problem will be solved if both try?

7. A, B, C have equal claims for a prize. A says to B, "You and C draw lots, and the winner shall draw lots with me for the prize." Is this fair?

8. A bag contains 5 tickets numbered 1, 2, 3, 4, 5. Three tickets are drawn at random, the tickets not being replaced after drawing. Find the chance of drawing a total of 10.

9. A bag contains 10 tickets, 5 marked 1, 2, 3, 4, 5, and 5 blank. Three tickets are drawn at random, each being replaced before the next is drawn. Find the probability of drawing a total of 10.

10. Find the probability of drawing in the previous example a total of 10 when the tickets are not replaced.

11. A bag contains four $10 gold-pieces, and six silver dollars. A person is entitled to draw 2 coins at random. Find the value of his expectation.

12. Six $5 pieces, four $3 pieces, and five coins which are either all gold dollars or all silver dimes are thrown together into a bag. Assuming that the unknown coins are equally likely to be dimes or dollars, what is a fair price to pay for the privilege of drawing at random a single coin?

13. A bag contains six $5 pieces, and four other coins which have all the same value. The expectation of drawing at random 2 coins is worth $8.40. Find the value of each of the unknown coins.

14. Find the probability of throwing at least one ace in 4 throws with a single die.

15. A copper is tossed 3 times. Find the chance that it will fall heads once and tails twice.

16. What is the chance of throwing double sixes at least once in 3 throws with a pair of dice?

17. Two bags contain each 4 black and 3 white balls. A ball is drawn at random from the first bag, and if it be white, it is put into the second bag, and a ball drawn at random from that bag. Find the odds against drawing two white balls.

18. A and B play at chess, and A wins on an average 2 games out of 3. Find the chance of A's winning exactly 4 games out of the first 6, drawn games being disregarded.

19. At tennis A on an average beats B 2 games out of 3. If they play one set, find the chance that A will win by the score of 6 to 2.

20. A and B, two players of equal skill, are playing tennis. A wants 2 games to complete the set, and B wants 3 games. Find the chance that A will win the set.

21. If n coins are tossed up, what is the chance that one, and only one, will turn up head?

22. A bag contains n balls. A person takes out one ball, and then replaces it. He does this n times. What is the chance that he has had in his hand every ball in the bag?

23. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected at least 3 will safely return?

24. At tennis A beats B on an average 2 games out of 3; if the score is 4 games to 3 in B's favor, find the chance of A's winning 6 games before B does.

25. A bets B \$10 to \$1 that he will throw heads at least once in 3 trials. What is B's expectation? What would have been a fair bet?

26. A draws 5 times (replacing) from a bag containing 3 white and 7 black balls, drawing each time one ball; every time he draws a white ball he is to receive \$1, and every time he draws a black ball he is to pay 50 cents. What is his expectation?

27. From a bag containing 2 eagles, 3 dollars, and 3 quarter-dollars, A is to draw 1 coin and then B 3 coins; and A, B, and C are to divide equally the value of the remainder. What are their expectations?

330.* Existence of Causes. In the problems thus far considered we have been concerned only with future events; we now proceed to a different class of problems, problems of which the following is the general type.

An event has happened. There are several possible causes, of which one *must* have existed, and only one *can* have existed. From the several possible causes a particular cause is selected; required the chance that this was the true cause.

Before proceeding to the general problem we shall consider some examples.

(1) Ten has been thrown with 2 dice. Required the chance that the throw was double fives.

Ten can be thrown in 3 ways: 6, 4; 4, 6; 5, 5. One of these three ways must have occurred, and only one can have occurred.

Before the event the chances that these respective ways *would* occur were all equal.

We shall *assume* that *after the event* the chances that these respective ways *have* occurred are all equal.

Then, precisely as in § 315, the chance that the throw was double fives is $\frac{1}{3}$; and the chance that the throw was a six and a four is $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$.

(2) Fifteen has been thrown with 3 dice. Required the chance that the throw was 3 fives.

Fifteen can be thrown in 10 ways:

6 5 4	5 4 6	4 5 6	6 6 3	3 6 6
6 4 5	5 6 4	4 6 5	6 3 6	5 5 5

One of these 10 ways must have occurred, and only one can have occurred.

Before the event the chances that these respective ways *would* occur were all equal.

We shall *assume* that *after the event* the chances that the respective ways *have* occurred are all equal.

Then, precisely as in § 315, the chance that the throw was 3 fives is $\frac{1}{10}$. *Ans.*

(3) A box contains 4 white balls and 2 black balls. Two balls are drawn at random and put into a second box. From the second box 1 ball is then drawn and found to be white. Required the chance that the two balls in the second box are both white.

Before the event there were three cases which might exist. These cases, with the respective chances of existence, were as follows:

The second box might contain:

(a) 2 white balls, of which the chance was $\frac{2}{5}$.
(b) 1 white and 1 black ball, of which the chance was $\frac{8}{15}$.
(c) 2 black balls, of which the chance was $\frac{1}{15}$.

Since 1 white ball has been drawn, (c) is impossible; we have, therefore, only (a) and (b) to consider.

Supposing (a) to exist, the chance of drawing a white ball from the second box was 1; supposing (b) to exist, the chance of drawing a white ball from the second box was $\frac{1}{2}$.

Hence, the chance *before the event* that (b) exists, and we draw a white ball, that is, the chance that we draw a white ball from two white balls, was $\frac{2}{5} \times 1 = \frac{2}{5}$; the chance *before the event* that (b) exists, and we draw a white ball, that is, the chance that we draw a white ball from a white and a black ball, was $\frac{8}{15} \times \frac{1}{2} = \frac{4}{15}$.

Represent by Q_1 the chance *after the event* that (a) existed, and by Q_2 the chance *after the event* that (b) existed.

We shall *assume* that Q_1 and Q_2 are *proportional* to the chance *before the event* that a white ball would be drawn from (a), and the chance *before the event* that a white ball would be drawn from (b).

This assumption corresponds to the assumption in Examples (1) and (2), in which the cases were equally likely to occur. We assume, then, that

$$Q_1 : Q_2 = \tfrac{2}{5} : \tfrac{4}{15}, \text{ or } \frac{Q_1}{\frac{2}{5}} = \frac{Q_2}{\frac{4}{15}}.$$

$$\therefore\ \frac{Q_1}{\frac{2}{5}} = \frac{Q_2}{\frac{4}{15}} = \frac{Q_1 + Q_2}{\frac{2}{5} + \frac{4}{15}}.$$

But $Q_1 + Q_2 = 1$, since either (a) or (b) *must* exist; also $\frac{2}{5} + \frac{4}{15} = \frac{2}{3}$.

$$\therefore \frac{Q_1}{\frac{2}{5}} = \frac{Q_2}{\frac{4}{15}} = \frac{1}{\frac{2}{3}}.$$

$$\therefore Q_1 = \tfrac{3}{5}, \text{ and } Q_2 = \tfrac{2}{5}.$$

The chance that both balls are white is $\frac{3}{5}$. *Ans.*

331.* In general, let P_1, P_2, P_3, ..., be the chance *before the event* that the first, second, third, ..., cause exists; and p_1, p_2, p_3, ..., the chance *before the event* that, when the first, second, third, ..., cause exists, the event will follow. Let Q_1, Q_2, Q_3, ..., be the chance *after the event* that the first, second, third, ..., cause existed.

Then P_1p_1 is the chance before the event that the event will happen from the first cause; P_2p_2, the chance before the event that the event will happen from the second cause; and so on.

We shall *assume* that Q_1, Q_2, Q_3, ..., are respectively proportional to P_1p_1, P_2p_2, P_3p_3, ... ;

that is, $$\frac{Q_1}{P_1p_1} = \frac{Q_2}{P_2p_2} = \frac{Q_3}{P_3p_3} = \cdots\cdots$$

Therefore, by § 93,

$$\frac{Q_1}{P_1p_1} = \frac{Q_2}{P_2p_2} = \frac{Q_3}{P_3p_3} = \cdots\cdots \frac{Q_1 + Q_2 + Q_3 + \cdots\cdots}{P_1p_1 + P_2p_2 + P_3p_3 + \cdots\cdots}.$$

But $Q_1 + Q_2 + Q_3 + \cdots\cdots = 1$, since some one of the causes must exist. Hence,

$$\frac{Q_1}{P_1p_1} = \frac{Q_2}{P_2p_2} = \frac{Q_3}{P_3p_3} = \cdots\cdots \frac{1}{P_1p_1 + P_2p_2 + P_3p_3 + \cdots\cdots},$$

from which Q_1, Q_2, Q_3, ..., may be readily found.

Exercise 51.*

1. An even number greater than 6 has been thrown with 2 dice. What is the chance that doublets were thrown?

2. A number divisible by 3 has been thrown with 2 dice. What is the chance that the number was odd?

3. Fourteen has been thrown with 3 dice. Find the chance that one and only one of the dice turned up a six.

4. An even number greater than 10 has been thrown with 3 dice. Find the chance that the number was 14.

5. From a bag containing 6 white and 2 black balls a person draws 3 balls at random and places them in a second bag. A second person then draws from the second bag 2 balls and finds them to be both white. Find the chance that the third ball in the second bag is white.

6. A bag contains 4 balls, each of which is equally likely to be white or black. A person is to receive $12 if all four are white. Find the value of his expectation.

Suppose he draws 2 balls and finds them to be both white. What is now the value of his expectation?

7. A and B obtain the same answer to a certain problem. It is found that A obtains a correct answer 11 times out of 12, and B 9 times out of 10. If it is 100 to 1 against their making the same mistake, find the chance that the answer they both obtain is correct.

8. From a pack of 52 cards one has been lost; from the imperfect pack 2 cards are drawn and found to be both spades. Required the chance that the missing card is a spade.

9. A speaks truth 9 times out of 10, and B 11 times out of 12. There is a certain event which must either happen or fail, and is of itself twice as likely to happen as to fail. A says that the event happened, and B that it failed. Find the odds for the event happening.

CHAPTER XXIII.

CONTINUED FRACTIONS.

332. A fraction in the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\text{etc.}}}}$$

is called a **continued fraction**, though the term is commonly restricted to a continued fraction that has 1 for each of its numerators, as

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

We shall consider in this chapter some of the elementary properties of such fractions.

333. *Any proper fraction in its lowest terms may be converted into a terminated continued fraction.*

Let $\frac{b}{a}$ be such a fraction; then, if p be the quotient and c the remainder of $a \div b$,

$$\frac{b}{a}=\frac{1}{\frac{a}{b}}=\frac{1}{p+\frac{c}{b}};$$

if q be the quotient and d the remainder of $b \div c$,

$$\frac{1}{p+\frac{c}{b}}=\frac{1}{p+\frac{1}{\frac{b}{c}}}=\cfrac{1}{p+\cfrac{1}{q+\frac{d}{c}}}.$$

Hence, $$\frac{b}{a}=\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}}$$

The successive steps of the process are the same as the steps for finding the H. C. F. of a and b; and since a and b are prime to each other, a remainder, 1, will at length be reached, and the fraction terminates.

Observe that p, q, r, ..., are all positive integers.

334. Convergents. The fractions formed by taking one, two, three, ..., of the quotients p, q, r, ..., are

$$\frac{1}{p},\quad \cfrac{1}{p+\cfrac{1}{q}},\quad \cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r}}},\ \ldots\ldots,$$

which simplified are

$$\frac{1}{p},\quad \frac{q}{pq+1},\quad \frac{qr+1}{(pq+1)r+p},\ \ldots\ldots,$$

and are called the first, second, and third convergents, respectively.

335. *The successive convergents are alternately greater than and less than the true value of the given fraction.*

Let x be the true value of

$$\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}};$$

then, since p, q, r, ..., are positive integers,

$$p<p+\cfrac{1}{q+\cfrac{1}{r+\text{etc.}}}$$

$$\therefore \frac{1}{p} > \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r} + \text{etc.}}}; \text{ that is, } \frac{1}{p} > x.$$

Again, $$q < q + \frac{1}{r} + \text{etc.}$$

$$\therefore \frac{1}{q} > \cfrac{1}{q + \cfrac{1}{r} + \text{etc.}}$$

$$\therefore \cfrac{1}{p + \cfrac{1}{q}} < \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r} + \text{etc.}}};$$

that is, $$\cfrac{1}{p + \cfrac{1}{q}} < x; \text{ and so on.}$$

336. *If* $\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}$ *are any three consecutive convergents, and if* m_1, m_2, m_3 *be the quotients that produced them, then* $\frac{u_3}{v_3} = \frac{m_3u_2 + u_1}{m_3v_2 + v_1}$.

For, if the first three quotients are p, q, r, the first three convergents are (§ 334),

$$\frac{1}{p}, \frac{q}{pq+1}, \frac{qr+1}{(pq+1)r+p} \cdots\cdots \quad (1)$$

From (§ 334) it is seen that the second convergent is formed from the first by writing in it $p + \frac{1}{q}$ for p; and the third from the second by writing $q + \frac{1}{r}$ for q. In this way, any convergent may be formed from the preceding convergent.

Therefore, $\frac{u_3}{v_3}$ will be formed from $\frac{u_2}{v_2}$ by writing $m_2+\frac{1}{m_3}$ for m_2.

In (1) it is seen that the third convergent has its numerator $= r \times$ (second numerator) + (first numerator); and its denominator $= r \times$ (second denominator) + (first denominator).

Assume that this law holds true for the third of the three consecutive convergents

$$\frac{u_0}{v_0},\ \frac{u_1}{v_1},\ \frac{u_2}{v_2},\ \text{so that,}\ \frac{u_2}{v_2}=\frac{m_2u_1+u_0}{m_2v_1+v_0}. \tag{2}$$

Then, since $\frac{u_3}{v_3}$ is formed from $\frac{u_2}{v_2}$ by using $m_2+\frac{1}{m_3}$ for m_2,

$$\frac{u_3}{v_3}=\frac{\left(m_2+\frac{1}{m_3}\right)u_1+u_0}{\left(m_2+\frac{1}{m_3}\right)v_1+v_0}=\frac{m_3(m_2u_1+u_0)+u_1}{m_3(m_2v_1+v_0)+v_1}.$$

Substitute u_2 and v_2 for their values $m_2u_1+u_0$ and $m_2v_1+v_0$; then

$$\frac{u_3}{v_3}=\frac{m_3u_2+u_1}{m_3v_2+v_1}.$$

Therefore the law still holds true; and as it has been shown to be true for the third convergent, the law is general. (Note on p. 201.)

337. *The difference between two consecutive convergents,* $\frac{u_1}{v_1}$ *and* $\frac{u_2}{v_2}$ *is* $\frac{1}{v_1v_2}$.

The difference between the first two convergents is

$$\frac{1}{p}-\frac{q}{pq+1}=\frac{1}{p(pq+1)}.$$

Let the sign $\sim$ mean *the difference between*, and assume the proposition true for $\frac{u_0}{v_0}$ and $\frac{u_1}{v_1}$;

then $$\frac{u_0}{v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_0 v_1} = \frac{1}{v_0 v_1}.$$

But

$$\frac{u_2}{v_2} \sim \frac{u_1}{v_1} = \frac{u_2 v_1 \sim u_1 v_2}{v_1 v_2} = \frac{(m_2 u_1 + u_0) v_1 \sim u_1 (m_2 v_1 + v_0)}{v_1 v_2}$$

(substituting for u_2 and v_2 their values, $m_2 u_1 + u_0$ and $m_2 v_1 + v_0$).

Reducing, $$\frac{u_2}{v_2} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_1 v_2},$$

$$= \frac{1}{v_1 v_2} \text{ (by assumption).}$$

Hence, if the proposition be true for one pair of consecutive convergents, it will be true for the next pair; but it has been shown to be true for the *first* pair, therefore it is true for *every* pair. (Note on p. 201.)

Since by § 335 the true value of x lies between two consecutive convergents, $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$, the convergent $\frac{u_1}{v_1}$ will differ from x by a number less than $\frac{u_1}{v_1} \sim \frac{u_2}{v_2}$; that is, by a number less than $\frac{1}{v_1 v_2}$; so that the error in taking $\frac{u_1}{v_1}$ for x is less than $\frac{1}{v_1 v_2}$, and therefore less than $\frac{1}{{v_1}^2}$, as $v_2 > v_1$ since $v_2 = m_2 v_1 + v_0$.

Any convergent, $\frac{u_1}{v_1}$, is in its lowest terms; for, if u_1 and v_1 had any common factor, it would also be a factor of $u_1 v_2 \sim u_2 v_1$; that is, a factor of 1.

338. *The successive convergents approach more and more nearly to the true value of the continued fraction.*

Let $\frac{u_0}{v_0}$, $\frac{u_1}{v_1}$, $\frac{u_2}{v_2}$ be consecutive convergents.

Now $\frac{u_2}{v_2}$ differs from x, the true value of the fraction, only because m_2 is used instead of $m_2 + \frac{1}{m_3 + \text{etc.}}$.

Let this complete quotient, which is always greater than unity, be represented by M.

Then, since $\quad \frac{u_2}{v_2} = \frac{m_2 u_1 + u_0}{m_2 v_1 + v_0}, \quad x = \frac{M u_1 + u_0}{M v_1 + v_0}$.

$$\therefore \; x \sim \frac{u_1}{v_1} = \frac{M u_1 + u_0}{M v_1 + v_0} \sim \frac{u_1}{v_1} = \frac{u_0 v_1 \sim u_1 v_0}{v_1(M v_1 + v_0)} = \frac{1}{v_1(M v_1 + v_0)}.$$

And

$$\frac{u_0}{v_0} \sim x = \frac{u_0}{v_0} \sim \frac{M u_1 + u_0}{M v_1 + v_0} = \frac{M(u_0 v_1 \sim u_1 v_0)}{v_0(M v_1 + v_0)} = \frac{M}{v_0(M v_1 + v_0)}.$$

Now $1 < M$ and $v_1 > v_0$, and for both these reasons

$$x \sim \frac{u_1}{v_1} < \frac{u_0}{v_0} \sim x.$$

That is, $\frac{u_1}{v_1}$ is nearer to x than $\frac{u_0}{v_0}$ is.

339. *Any convergent* $\frac{u_1}{v_1}$ *is nearer the true value* x *than any other fraction with smaller denominator*

Let $\frac{a}{b}$ be a fraction in which $b < v_1$.

If $\frac{a}{b}$ is one of the convergents, $x \sim \frac{a}{b} > \frac{u_1}{v_1} \sim x$. § 338

If $\frac{a}{b}$ is not one of the convergents, and is nearer to x

than $\frac{u_1}{v_1}$ is, then, since x lies between $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ (§ 335), $\frac{a}{b}$ must be nearer to $\frac{u_2}{v_2}$ than $\frac{u_1}{v_1}$ is; that is,

$$\frac{a}{b} \sim \frac{u_2}{v_2} < \frac{u_1}{v_1} \sim \frac{u_2}{v_2}, \text{ or } \frac{v_2 a \sim u_2 b}{v_2 b} < \frac{1}{v_1 v_2};$$

and since $b < v_1$, this would require that $v_2 a \sim u_2 b < 1$; but $v_2 a \sim u_2 b$ cannot be less than 1, for a, b, u_2, v_2 are all integers. Hence, $\frac{u_1}{v_1}$ is nearer to x than $\frac{a}{b}$ is.

340. Find the continued fraction equal to $\frac{31}{75}$, and also the successive convergents.

Following the process of finding the H. C. F. of 31 and 75, the successive quotients are found to be 2, 2, 2, 1, 1, 2. Hence the continued fraction is

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}}$$

To find the successive convergents:

Write the successive quotients in line, $\frac{0}{1}$ under the first quotient, $\frac{1}{2}$ under the second quotient, and then (§ 336) multiply each term by the quotient above it, and add the term to the left to obtain the corresponding term to the right. Thus,

$$\begin{aligned} &\text{Quotients} &&= 2,\ 2,\ 2,\ 1,\ 1,\ 2. \\ &\text{Convergents} &&= \tfrac{0}{1},\ \tfrac{1}{2},\ \tfrac{2}{5},\ \tfrac{5}{12},\ \tfrac{7}{17},\ \tfrac{12}{29},\ \tfrac{31}{75}. \end{aligned}$$

It is convenient to begin to reckon with $\frac{0}{1}$, but the next convergent, in this case $\frac{1}{2}$, is called the *first* convergent.

NOTE. Continued fractions are often written in a more compact form. Thus, the above fraction may be written

$$\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$$

341. A quadratic surd may be expressed in the form of a *non-terminating* continued fraction.

To express $\sqrt{3}$ in the form of a continued fraction.

Suppose $$\sqrt{3} = 1 + \frac{1}{x} \text{ (as 1 is the greatest integer in } \sqrt{3});$$

then $$\frac{1}{x} = \sqrt{3} - 1.$$

$$\therefore x = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}.$$

Suppose $$\frac{\sqrt{3}+1}{2} = 1 + \frac{1}{y}\left(\text{as 1 is the greatest integer in } \frac{\sqrt{3}+1}{2}\right);$$

then $$\frac{1}{y} = \frac{\sqrt{3}+1}{2} - 1 = \frac{\sqrt{3}-1}{2}.$$

$$\therefore y = \frac{2}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{1}.$$

Suppose $$\frac{\sqrt{3}+1}{1} = 2 + \frac{1}{z}\left(\text{as 2 is the greatest integer in } \frac{\sqrt{3}+1}{1}\right):$$

then $$\frac{1}{z} = \frac{\sqrt{3}+1}{1} - 2 = \sqrt{3} - 1.$$

$$\therefore z = \frac{1}{\sqrt{3}-1}.$$

This is the same as x above; hence, the quotients 1, 2, will be continually repeated.

$$\therefore \sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2} + \text{etc.}}$$

of which $\cfrac{1}{1+\cfrac{1}{2}}$ will be continually repeated, and the whole expression may be written,

$$1 + \frac{\dot{1}}{1} + \frac{\dot{1}}{2}.$$

The convergents will be 1, 2, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}$, $\frac{71}{41}$, etc.

342. A continued fraction in which the denominators recur is called a **periodic** continued fraction.

The value of a periodic continued fraction can be expressed as the root of a quadratic equation.

Find the surd value of $\dot{\frac{1}{1}}+\dot{\frac{1}{2}}.$

Let x be the value;

then $$x=\cfrac{1}{1+\cfrac{1}{2+x}}=\frac{2+x}{3+x};$$

$$\therefore\ x^2+2x=2,$$

$$x=-1+\sqrt{3}.$$

We take the + sign since x is evidently positive.

343. An exponential equation can be solved by continued fractions.

Solve by continued fractions $10^x=2$.

Suppose $$x=0+\frac{1}{y};$$

then $$10^{\frac{1}{y}}=2,$$

or $$10=2^y.$$

$$\therefore\ y=3+\frac{1}{z}\ (\text{as 10 lies between } 2^3 \text{ and } 2^4).$$

Then $$10=2^{3+\frac{1}{z}}=2^3\times 2^{\frac{1}{z}};$$

or $$2^{\frac{1}{z}}=\tfrac{10}{8}=\tfrac{5}{4},$$

and $$2=(\tfrac{5}{4})^z.$$

$$\therefore\ z=3+\frac{1}{u}\ \left[\text{as 2 lies between } \left(\frac{5}{4}\right)^3 \text{ and } \left(\frac{5}{4}\right)^4\right]$$

Then $$2=(\tfrac{5}{4})^{3+\frac{1}{u}}=(\tfrac{5}{4})^3\times(\tfrac{5}{4})^{\frac{1}{u}};$$

or $$(\tfrac{5}{4})^{\frac{1}{u}}=\tfrac{128}{125},$$

and $$\tfrac{5}{4}=(\tfrac{128}{125})^u.$$

The greatest integer in u will be found to be 9.

Hence,
$$x = 0 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{9} + \text{etc.}}}$$

The successive convergents will be $\frac{1}{3}$, $\frac{3}{10}$, $\frac{28}{93}$, etc.

The last gives $\qquad x = \frac{28}{93} = 0.3010$, *nearly*.

Observe that by the above process we have calculated the common logarithm of 2. By § 337 the error, when 0.3010 is taken for the common logarithm of 2, is considerably less than $\frac{1}{(93)^2}$; that is considerably less than 0.00011; so that 0.3010 is certainly correct to three places of decimals, and probably correct to four places.

Logarithms are, however, much more easily calculated by the use of series, as will be shown in a following chapter.

Exercise 52.

1. Find the values of:

$$\frac{1}{4+}\frac{1}{3+}\frac{1}{2}; \quad \frac{1}{2+}\frac{1}{3+}\frac{1}{7}; \quad \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{4+}\frac{1}{5}.$$

2. Find continued fractions for $\frac{123}{157}$; $\frac{159}{47}$; $\frac{103}{71}$; $\frac{67}{177}$; $\sqrt{5}$; $\sqrt{11}$; $4\sqrt{6}$; and find the fourth convergent to each.

3. Find continued fractions for $\frac{47}{257}$; $\frac{457}{204}$; $\frac{2065}{4626}$; $\frac{2991}{568}$; and find the third convergent to each.

4. Find continued fractions for $\sqrt{21}$; $\sqrt{22}$; $\sqrt{33}$; $\sqrt{55}$.

5. Obtain convergents, with only two figures in the denominator, that approach nearest to the values of: $\sqrt{7}$; $\sqrt{10}$; $\sqrt{15}$; $\sqrt{17}$; $\sqrt{18}$; $\sqrt{20}$; $3-\sqrt{5}$; $2+\sqrt{11}$.

6. Find the proper fraction which, if converted into a continued fraction, will have quotients 1, 7, 5, 2.

7. Find the next convergent when the two preceding convergents are $\frac{3}{17}$ and $\frac{19}{89}$, and the next quotient is 5.

8. If the pound troy is the weight of 22.8157 cubic inches of water, and the pound avoirdupois of 27.7274 cubic inches of water, find a fraction with denominator less than 100 which shall differ from their ratio by less than 0.0001.

9. The ratio of the diagonal to a side of a square being $\sqrt{2}$, find a fraction with denominator less than 100 which shall differ from their ratio by less than 0.0001.

10. The ratio of the circumference of a circle to its diameter being approximately the ratio of 3.14159265 : 1, find the first three convergents to this ratio, and determine to how many decimal places each may be depended upon as agreeing with the true value.

11. In two scales of which the zero-points coincide the distances between consecutive divisions of the one are to the corresponding distances of the other as 1 : 1.06577. Find what division-points most nearly coincide.

12. Find the surd values of:

$$1+\frac{\dot{1}}{4+}\frac{\dot{1}}{2};\quad 3+\frac{\dot{1}}{1+}\frac{\dot{1}}{6};\quad \frac{\dot{1}}{3+}\frac{1}{1+}\frac{\dot{1}}{6};\quad 1+\frac{\dot{1}}{2+}\frac{1}{3+}\frac{\dot{1}}{4}.$$

13. Prove that

$$\left(a+\frac{\dot{1}}{b+}\frac{\dot{1}}{a}\right)\left(\frac{\dot{1}}{b+}\frac{\dot{1}}{a}\right)=\frac{a}{b}.$$

14. Show that the ratio of the diagonal of a cube to its edge may be nearly expressed by 97 : 56. Find the greatest possible value of the error made in taking this ratio for the true ratio.

15. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.

16. Find a series of fractions converging to the ratio of a cubic yard to a cubic meter, if a cubic yard is $\frac{76453}{100000}$ of a cubic meter.

CHAPTER XXIV.

SCALES OF NOTATION.

344. Definitions. Let any positive integer be selected as a **radix** or **base**; then any number may be expressed as a polynomial of which the terms are multiples of powers of the radix.

Any positive integer may be selected as the radix; and to each radix corresponds a **scale of notation**.

In writing the polynomials they are arranged by descending powers of the radix, and the powers of the radix are omitted, the *place* of each digit indicating of what power of the radix it is the coefficient.

Thus, in the scale of ten, 2356 stands for

$$2 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 6;$$

in the scale of seven for

$$2 \times 7^3 + 3 \times 7^2 + 5 \times 7 + 6;$$

in the scale of r for

$$2r^3 + 3r^2 + 5r + 6.$$

345. Computation. Computations are made with numbers in any scale, by observing that one unit of any order is equal to the radix-number of units of the next lower order; and that the radix-number of units of any order is equal to one unit of the next higher order.

(1) Add 56,432 and 15,646 (scale of seven).

$$\begin{array}{r} 56432 \\ 15646 \\ \hline 105411 \end{array}$$

The process differs from that in the decimal scale only in that when a sum greater than *seven* is reached, we divide by *seven* (not ten), set down the remainder, and carry the quotient to the next column.

(2) Subtract 34,561 from 61,235 (scale of eight).

```
 61235
 34561
 -----
 24454
```

We add eight, instead of ten as in the common scale.

(3) Multiply 5732 by 428 (scale of nine).

```
    5732
     428
   -----
   51477
  12564
 24238
 -------
 2612127
```

We divide each time by *nine*, set down the remainder, and carry the quotient.

(4) Divide 2,612,127 by 5732 (scale of nine).

```
5732)2612127(428
     24238
     -----
      17732
      12564
      -----
       51477
       51477
       -----
```

346. Integers in Any Scale. *If* **r** *be any positive integer, any positive integer* **N** *may be expressed in the form*

$$N = ar^n + br^{n-1} + \cdots + pr^2 + qr + s,$$

in which the coefficients **a**, **b**, **c**, ..., *are positive integers, each less than* **r**.

For, divide N by r^n, the highest power of r contained in N, and let the quotient be a with the remainder N_1.

Then, $N = ar^n + N_1$.

In like manner, $N_1 = br^{n-1} + N_2$; $N_2 = cr^{n-2} + N_3$; and so on.

By continuing this process, a remainder s will at length be reached which is less than r. So that,

$$N = ar^n + br^{n-1} + \cdots + pr^2 + qr + s.$$

Some of the coefficients s, q, p, ... may vanish, and each will be less than r; that is, their values may range from zero to $r-1$. Hence, including zero, r digits will be required to express numbers in the scale of r.

To express any positive integer **N** *in the scale of* **r**.

It is required to express N in the form

$$ar^n + br^{n-1} + \cdots + pr^2 + qr + s,$$

and to show how the digits a, b, ... may be found.

If $$N = ar^n + br^{n-1} + \cdots + pr^2 + qr + s,$$

then $$\frac{N}{r} = ar^{n-1} + br^{n-2} + \cdots + pr + q + \frac{s}{r}.$$

That is, the remainder on dividing N by r is s, the last digit.

Let $$N_1 = ar^{n-1} + br^{n-2} + \cdots + pr + q;$$

then $$\frac{N_1}{r} = ar^{n-2} + br^{n-3} + \cdots + p + \frac{q}{r}.$$

That is, the remainder is q, the last but one of the digits.

Hence, to express an integral number in a proposed scale,

Divide the number by the radix, then the quotient by the radix, and so on; the successive remainders will be the successive digits beginning with the units' place.

(1) Express 42,897 (scale of ten) in the scale of six.

$$\begin{array}{r l}
6\,)\underline{42897} & \\
6\,)\underline{7149} & \ldots 3 \\
6\,)\underline{1191} & \ldots 3 \\
6\,)\underline{198} & \ldots 3 \\
6\,)\underline{33} & \ldots 0 \\
5 & \ldots 3
\end{array}$$

Ans. 530,333.

(2) Change 37,214 from the scale of eight to the scale of nine.

$$\begin{array}{r l}
9)\underline{37214} & \\
9)\underline{3363} & \ldots 1 \\
9)\underline{305} & \ldots 6 \\
9)\underline{25} & \ldots 8 \\
2 & \ldots 3 \\
\textit{Ans.}\ 23{,}861. &
\end{array}$$

The radix is 8; and hence the two digits on the left, 37, do not mean *thirty-seven*, but $3 \times 8 + 7$, or *thirty-one*, which contains 9 three times, with the remainder 4.

The next partial dividend is $4 \times 8 + 2 = 34$, which contains 9 three times, with the remainder 7; and so on.

(3) In what scale is 140 (scale of ten) expressed by 352?

Let r be the radix; then, in the scale of ten,

$$140 = 3r^2 + 5r + 2 \quad \text{or} \quad 3r^2 + 5r = 138.$$

Solving, we find $r = 6$.

The other value of r is fractional, and therefore inadmissible, since the radix is always a positive integer.

347. Radix-Fractions. As in the decimal scale decimal fractions are used, so in any scale *radix-fractions* are used.

Thus, in the decimal scale, 0.2341 stands for

$$\frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{1}{10^4};$$

and in the scale of r it stands for

$$\frac{2}{r} + \frac{3}{r^2} + \frac{4}{r^3} + \frac{1}{r^4}.$$

(1) Express $\frac{245}{256}$ (scale of ten) by a radix-fraction in the scale of eight.

Assume $$\frac{245}{256} = \frac{a}{8} + \frac{b}{8^2} + \frac{c}{8^3} + \frac{d}{8^4} + \cdots\cdots$$

Multiply by 8, $$7\tfrac{21}{32} = a + \frac{b}{8} + \frac{c}{8^2} + \frac{d}{8^3} + \cdots\cdots$$

$\therefore\ a = 7$, and $$\frac{21}{32} = \frac{b}{8} + \frac{c}{8^2} + \frac{d}{8^3} + \cdots\cdots$$

Multiply by 8, $5\frac{1}{4} = b + \frac{c}{8} + \frac{d}{8^2} + \cdots$

$\therefore\ b = 5$, and $\frac{1}{4} = \frac{c}{8} + \frac{d}{8^2} + \cdots$

Multiply by 8, $2 = c + \frac{d}{8} + \cdots$

$\therefore\ c = 2$, and $0 = d$, etc.

The answer is 0.752.

(2) Change 35.14 from the scale of eight to the scale of six.

We take the integral and fractional parts separately.

Integral part: $6\underline{)35}$
$\quad 4 \quad 5.$

Fractional part:

$$\frac{1}{8} + \frac{4}{8^2} = \frac{12}{64} = \frac{3}{16}.$$

This is reduced to a radix-fraction in the scale of six as follows:

$$\begin{array}{r}
3 \\
\underline{6} \\
16)\overline{18}(1 \\
\underline{16} \\
2 \\
\underline{6} \\
16)\overline{12}(0 \\
\underline{6} \\
16)\overline{72}(4 \\
\underline{64} \\
8 \\
\underline{6} \\
16)\overline{48}(3 \\
\underline{48}
\end{array}$$

45.1043. *Ans.*

Exercise 53.

1. Add together 435, 624, 737 (scale of eight).
2. From 32,413 subtract 15,542 (scale of six).
3. Multiply 6431 by 35 (scale of seven).
4. Multiply 4685 by 3483 (scale of nine).
5. Divide 102,432 by 36 (scale of seven).
6. Find H. C. F. of 2541 and 3102 (scale of seven).
7. Extract the square root of 33,224 (scale of six).

8. Extract the square root of 300,114 (scale of five).

9. Change 624 from the scale of ten to the scale of five.

10. Change 3516 from the scale of seven to the scale of ten.

11. Change 3721 from the scale of eight to the scale of six.

12. Change 4535 from the scale of seven to the scale of nine.

13. Change 32.15 from the scale of six to the scale of nine.

14. Express $\frac{25}{128}$ (scale of ten) by a radix-fraction in the scale of four.

15. Express $\frac{43}{108}$ (scale of ten) by a radix-fraction in the scale of six.

16. Multiply 31.24 by 0.31 (scale of five).

17. In what scale is this true? $21 \times 36 = 746$.

18. In what scale is the square of 23 expressed by 540?

19. In what scale are 212, 1101, 1220 in arithmetical progression?

20. Show that 1,234,321 is a perfect square in any scale (radix greater than four).

21. Which of the weights 1, 2, 4, 8, ... pounds must be selected to weigh 345 pounds, only one weight of each kind being used?

22. If two numbers are formed by the same digits in different orders, prove that the difference of the numbers is divisible by $r - 1$.

CHAPTER XXV.

THEORY OF NUMBERS.

348. Definitions. In the present chapter, by *number* will be meant *positive integer*. The terms *prime, composite,* will be used in the ordinary arithmetical sense.

A *multiple* of a is a number which contains the factor a, and may be written ma.

An even number, since it contains the factor 2, may be written $2m$; an odd number may be written $2m+1$, $2m-1$, $2m+3$, $2m-3$, etc.

A number a is said *to divide* another number b when $\frac{b}{a}$ is an integer.

349. Resolution into Prime Factors. *A number can be resolved into prime factors in only one way.*

Let N be the number; suppose $N = abc \ldots\ldots$, where $a, b, c, \ldots\ldots$ are prime numbers; suppose also $N = \alpha\beta\gamma \ldots\ldots$ where $\alpha, \beta, \gamma, \ldots\ldots$ are prime numbers.

Then, $$abc \cdots\cdots = \alpha\beta\gamma \cdots\cdots$$

Hence, α must divide the product $abc \ldots\ldots$; but $a, b, c, \ldots\ldots$ are all prime numbers; hence α must be equal to some one of them, a suppose.

Dividing by a, $$bc \cdots\cdots = \beta\gamma \cdots\cdots,$$

and so on. Hence, the factors in $\alpha\beta\gamma \ldots\ldots$ are equal to those in $abc \ldots\ldots$, and the theorem is proved.

350. Divisibility of a Product. I. *If a number* **a** *divides a product* **bc**, *and is prime to* **b**, *it must divide* **c**.

For, since a divides bc, every prime factor of a must be found in bc; but since a is prime to b, no factor of a will be found in b; hence all the prime factors of a are found in c; that is, a divides c.

From this theorem it follows that:

II. *If a prime number* $\mathbf{a}$ *divides a product* $\mathbf{bcde}$....., *it must divide some factor of that product; and conversely.*

III. *If a prime number divides* $\mathbf{b}^n$, *it must divide* $\mathbf{b}$.

IV. *If* $\mathbf{a}$ *is prime to* $\mathbf{b}$ *and* $\mathbf{c}$, *it is prime to* $\mathbf{bc}$.

V. *If* $\mathbf{a}$ *is prime to* $\mathbf{b}$, *every power of* $\mathbf{a}$ *is prime to every power of* $\mathbf{b}$.

351. Theorem. *If* $\frac{\mathbf{a}}{\mathbf{b}}$, *a fraction in its lowest terms, is equal to another fraction* $\frac{\mathbf{c}}{\mathbf{d}}$, *then* $\mathbf{c}$ *and* $\mathbf{d}$ *are equimultiples of* $\mathbf{a}$ *and* $\mathbf{b}$.

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{ad}{b}=c$. Since b will not divide a, it must divide d; hence d is a multiple of b.

Let $d=mb$, m being an integer; since $\frac{a}{b}=\frac{c}{d}$, and $d=mb$, $\frac{a}{b}=\frac{c}{mb}$; therefore $c=ma$.

Hence, c and d are equimultiples of a and b.

From the above theorem, it follows that in the decimal scale of notation a common fraction in its lowest terms will produce a non-terminating decimal if its denominator contains any prime factor except 2 and 5.

For a terminating decimal is equivalent to a fraction with a denominator 10^n. Therefore, a fraction $\frac{a}{b}$ in its lowest terms cannot be equal to such a fraction, unless 10^n is a multiple of b. But 10^n, that is, $2^n \times 5^n$, contains no factors

besides 2 and 5, and hence cannot be a multiple of b, if b contains any factors except these.

352. Square Numbers. If a square number is resolved into its prime factors, the exponent of each factor will be even.

For, if $$N = a^p \times b^q \times c^r \cdots\cdots,$$
$$N^2 = a^{2p} \times b^{2q} \times c^{2r} \cdots\cdots$$

Conversely: A number which has the exponents of all its prime factors even will be a perfect square; therefore, to change any number to a perfect square,

Resolve the number into its prime factors, select the factors which have odd exponents, and multiply the given number by the product of these factors.

Thus, to find the least number by which 250 must be multiplied to make it a perfect square.

$250 = 2 \times 5^3$, in which 2 and 5 are the factors which have odd exponents.

Hence the multiplier required is $2 \times 5 = 10$.

353. Divisibility of Numbers.

I. *If two numbers* **N** *and* **N'**, *when divided by* **a**, *have the same remainder, their difference is divisible by* **a**.

For, if N when divided by a have a quotient q and a remainder r, then
$$N = qa + r.$$

And if N' when divided by a have a quotient q' and a remainder r, then
$$N' = q'a + r.$$

Therefore, $$N - N' = (q - q')a.$$

II. *If the difference of two numbers* **N** *and* **N'** *is divisible by* **a**, *then* **N** *and* **N'** *when divided by* **a** *will have the same remainder.*

For, if $$N - N' = (q - q')a,$$

then $$\frac{N}{a} - \frac{N'}{a} = q - q'.$$

Therefore, $$\frac{N}{a} - q = \frac{N'}{a} - q'.$$

That is, $$N - aq = N' - aq'.$$

III. *If two numbers* **N** *and* **N'**, *when divided by a given number* **a**, *have remainders* **r** *and* **r'**, *then* **NN'** *and* **rr'** *when divided by* **a** *will have the same remainder.*

For, if $$N = qa + r,$$

and $$N' = q'a + r',$$

then $$NN' = qq'a^2 + qar' + q'ar + rr'$$
$$= (qq'a + qr' + q'r)a + rr'.$$

Therefore, by II., NN' and rr' when divided by a will have the same remainder.

As a particular case, 37 and 47 when divided by 7 have remainders 2 and 5 respectively.

Now $37 \times 47 = 1739$ and $2 \times 5 = 10$.

The remainder, when each of these two numbers is divided by 7, is 3.

NOTE. From II. it follows that, in the scale of ten.

(1) A number is divisible by 2, 4, 8, ... if the numbers denoted by its last digit, last two digits, last three digits, ... are divisible respectively by 2, 4, 8, ...

(2) A number is divisible by 5, 25, 125, ... if the numbers denoted by its last digit, last two digits, last three digits, ... are divisible respectively by 5, 25, 125, ...

(3) If from a number the sum of its digits is subtracted, the remainder will be divisible by 9.

For, if from a number expressed in the form

$$a + 10b + 10^2c + 10^3d + \cdots\cdots$$

$$a + \quad b + \quad c + \quad d + \cdots\cdots \qquad \text{is subtracted,}$$

the remainder will be $(10-1)b + (10^2-1)c + (10^3-1)d + \cdots\cdots$ and $10-1$, 10^2-1, 10^3-1, ... will be a series of 9's.

Therefore, the remainder is divisible by 9.

(4) Hence, a number N may be expressed in the form

$$9n + s \text{ (if } s \text{ denotes the sum of its digits);}$$

and N will be divisible by 3 if s is divisible by 3; and also by 9 if s is divisible by 9.

(5) A number will be divisible by 11 if the difference between the sum of its digits in the even places and the sum of its digits in the odd places is 0 or a multiple of 11.

For, a number N expressed by digits (beginning from the right) $a, b, c, d, \ldots\ldots$ may be put in the form of

$$N = a + 10b + 10^2c + 10^3d + \cdots\cdots$$

$$\therefore\ N - a + b - c + d - \cdots\cdots = (10+1)b + (10^2-1)c + (10^3+1)d + \cdots\cdots$$

But $10+1$ is a factor of $10+1$, 10^2-1, 10^3+1, ...

Therefore, $N - a + b - c + d - \cdots\cdots$ is divisible by $10+1 = 11$.

Hence, the number N may be expressed in the form

$$11n + (a + c + \cdots\cdots) - (b + d + \cdots\cdots),$$

and will be a multiple of 11 if $(a + c + \cdots\cdots) - (b + d + \cdots\cdots)$ is 0 or a multiple of 11.

354. Theorem. *The product of r consecutive integers is divisible by* $\lfloor r$.

Represent by $P_{n,k}$ the product of k consecutive integers beginning with n.

Then, $\quad P_{n,k} = n(n+1) \cdots\cdots (n+k-1);$

$$\begin{aligned} P_{n+1,k+1} &= (n+1)(n+2) \cdots\cdots (n+k)(n+k+1) \\ &= n(n+1)(n+2) \cdots\cdots (n+k) \\ &\qquad + (k+1)(n+1)(n+2) \cdots\cdots (n+k). \end{aligned}$$

$$\therefore\ P_{n+1,k+1} = P_{n,k+1} + (k+1)P_{n+1,k}.$$

Assume, for the moment, that the product of any k consecutive integers is divisible by $\underline{|k}$.

Then, $\qquad P_{n+1,\,k+1} = P_{n,\,k+1} + (k+1)M\underline{|k}$;

or, $\qquad P_{n+1,\,k+1} = P_{n,\,k+1} + M\underline{|k+1}$;

where M is an integer.

From this it is seen that if $P_{n,\,k+1}$ is divisible by $\underline{|k+1}$, $P_{n+1,\,k+1}$ is also divisible by $\underline{|k+1}$; but $P_{1,\,k+1}$ is divisible by $\underline{|k+1}$ since $P_{1,\,k+1} = \underline{|k+1}$. $\therefore$ $P_{2,\,k+1}$ is divisible by $\underline{|k+1}$; $\therefore$ $P_{3,\,k+1}$ is divisible by $\underline{|k+1}$; and so on.

Hence, the product of any $k+1$ consecutive integers is divisible by $\underline{|k+1}$, if it is known that the product of any k consecutive integers is divisible by $\underline{|k}$. But the product of any 2 consecutive integers is divisible by $\underline{|2}$; therefore, the product of any 3 consecutive integers is divisible by $\underline{|3}$; therefore, the product of any 4 consecutive integers is divisible by $\underline{|4}$; and so on. Therefore, the product of any r consecutive integers is divisible by $\underline{|r}$.

355. Examples. (1) Show that every square number is of one of the forms $5n$, $5n-1$, $5n+1$.

Every number is of one of the forms:

$$5m-2,\ 5m-1,\ 5m,\ 5m+1,\ 5m+2.$$

$$(5m \pm 2)^2 = 25m^2 \pm 20m + 4 = 5(5m^2 \pm 4m + 1) - 1;$$

$$(5m \pm 1)^2 = 25m^2 \pm 10m + 1 = 5(5m^2 \pm 2m) + 1;$$

$$(5m)^2 = 25m^2 \qquad = 5(5m^2).$$

$\therefore$ every square number is of one of the three forms:

$$5n,\ 5n-1,\ 5n+1.$$

Hence, in the scale of ten, every square number must end in 0, 1, 4, 5, 6, or 9.

(2) Show that $n^5 - n$ is divisible by 30 if n is even.

$$\begin{aligned} n^5 - n &= n(n-1)(n+1)(n^2+1) \\ &= n(n-1)(n+1)(n^2-4+5) \\ &= n(n-1)(n+1)[(n-2)(n+2)+5]. \end{aligned}$$

$n(n-1)(n+1)$ is divisible by $\lfloor\underline{3}$ (§ 354).

One of the five consecutive numbers

$$n-2,\ n-1,\ n,\ n+1,\ n+2,$$

is divisible by 5, and n^5-n is therefore divisible by 5.

Hence n^5-n is divisible by $5\lfloor\underline{3}$, that is by 30.

Exercise 54.*

Find the least number by which each of the following numbers must be multiplied in order that the product may be a square number.

1. 2625. **2.** 3675. **3.** 4374. **4.** 74088.

5. If m and n are positive integers, both odd or both even, show that m^2-n^2 is divisible by 4.

6. Show that n^2-n is always even.

7. Show that n^3-n is divisible by 6 if n is even; and by 24 if n is odd.

8. Show that n^5-n is divisible by 240 if n is odd.

9. Show that n^7-n is divisible by 42 if n is even; and by 168 if n is odd.

10. Show that $n(n+1)(n+5)$ is divisible by 6.

11. Show that every cube number is of one of the forms, $9n,\ 9n-1,\ 9n+1$.

12. Show that every cube number is of one of the forms, $7n,\ 7n-1,\ 7n+1$.

13. Show that every number which is both a square and a cube is of the form $7n$ or $7n+1$.

14. Show that in the scale of ten every perfect fourth power ends in one of the figures 0, 1, 5, 6.

CHAPTER XXVI.

VARIABLES AND LIMITS.

356. Constants and Variables. A number that, under the conditions of the problem into which it enters, may be made to assume any one of an unlimited number of values is called a **variable.**

A number that, under the conditions of the problem into which it enters, has a fixed value is called a **constant.**

Variables are generally represented by x, y, z, etc.; constants, by the Arabic numerals, and by a, b, c, etc.

357. Functions. Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case one variable is said to be a **function** of the other.

Thus, if a man walks on a straight road at a uniform rate of a miles per hour, the number of miles he walks and the number of hours he walks are both variables, and the first is a function of the second. If y be the number of miles he has walked at the end of x hours, y and x are connected by the relation $y = ax$, and y is a function of x. Also $x = \frac{y}{a}$; hence, x is also a function of y.

When one of two variables is a function of the other, the relation between them is generally expressed by an equation. If a value of one variable is assumed, the corresponding value of the other variable can be found from the given equation of relation between the two variables.

The variable of which the value is assumed is called the *independent* variable; the variable of which the value is

found from the given relation of the two variables is called the *dependent* variable.

In the last example we may assume values of x, and find the corresponding values of y from the relation $y = ax$; or assume values of y, and find the corresponding values of x from the relation $x = \frac{y}{a}$. In the first case x is the independent variable, and y the dependent; in the second case y is the independent variable, and x the dependent.

358. Limits. As a variable changes its value, it may approach some constant; if the variable can be made to approach the constant *as near as we please*, but cannot be made *absolutely equal to the constant*, the variable is said to approach the constant *as a limit*, and the constant is called the **limit** of the variable.

Let x represent the sum of n terms of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots;$$

then (§ 227), $$x = \frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \frac{2^n - 1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$$

Suppose n to increase; then, $\frac{1}{2^{n-1}}$ decreases, and x approaches 2.

Since we can take as many terms of the series as we please, n can be made as large as we please; therefore, $\frac{1}{2^{n-1}}$ can be made as small as we please, and x can be made to approach 2 as near as we please.

We cannot, however, make x absolutely equal to 2.

If we take any *assigned* value, as $\frac{1}{10000}$, we can make the difference between 2 and x less than this assigned value; for we have only to take n so large that $\frac{1}{2^{n-1}}$ is less than $\frac{1}{10000}$; that is, that 2^{n-1} is greater than 10,000: this will be accomplished by taking n as large as 15. Similarly, by taking n large enough, we can make the difference between 2 and x less than *any* assigned value.

Since $2 - x$ can be made as small as we please, it follows that the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, as n is constantly increased, approaches 2 *as a limit*.

359. Test for a Limit. In order to prove that a variable approaches a constant as a limit, it is *necessary* and *sufficient* to prove that the difference between the variable and the constant can be made *as near to zero as we please*, but cannot be made *absolutely equal to zero*.

A variable may approach a constant without approaching it *as a limit*. Thus, in the last example x approaches 3, but not as a limit; for $3 - x$ cannot be made as near to 0 as we please, since it cannot be made less than 1.

360. Infinites. As a variable changes its value, it may constantly increase in numerical value; if the variable can become numerically greater than any assigned value, *however great* this assigned value may be, the variable is said to *increase without limit*, or to *increase indefinitely*.

When a variable is conceived to have a value greater than any assigned value, however great this assigned value may be, the variable is said to become *infinite;* such a variable is called an *infinite* number, or simply an **infinite.**

361. Infinitesimals. As a variable changes its value, it may constantly decrease in numerical value; if the variable can become numerically less than any assigned value, *however small* this assigned value may be, the variable is said to *decrease without limit*, or to *decrease indefinitely*.

In this case the variable approaches 0 as a limit.

When a variable which approaches 0 as a limit is conceived to have a value less than any assigned value, however small this assigned value may be, the variable is said to become *infinitesimal;* such a variable is called an *infinitesimal number*, or simply an **infinitesimal.**

362. Infinites and infinitesimals are *variables*, not constants. There is no idea of *fixed* value implied in either an infinite or an infinitesimal.

An infinitesimal is not 0. An infinitesimal is a variable arising from the division of a quantity into a constantly increasing number of parts; 0 is a constant arising from taking the difference of two equal quantities.

A number which cannot become infinite is said to be **finite.**

363. Relations between Infinites and Infinitesimals.

I. *If* $\mathbf{x}$ *is infinitesimal, and* $\mathbf{a}$ *is finite and not* 0, *then* $\mathbf{ax}$ *is infinitesimal.* For, ax can be made as small as we please since x can be made as small as we please.

II. *If* $\mathbf{X}$ *is infinite, and* $\mathbf{a}$ *is finite and not* 0, *then* $\mathbf{aX}$ *is infinite.* For aX can be made as large as we please since X can be made as large as we please.

III. *If* $\mathbf{x}$ *is infinitesimal, and* $\mathbf{a}$ *is finite and not* 0, *then* $\frac{\mathbf{a}}{\mathbf{x}}$ *is infinite.* For $\frac{a}{x}$ can be made as large as we please since x can be made as small as we please.

IV. *If* $\mathbf{X}$ *is infinite, and* $\mathbf{a}$ *is finite and not* 0, *then* $\frac{\mathbf{a}}{\mathbf{X}}$ *is infinitesimal.* For $\frac{a}{X}$ can be made as small as we please since X can be made as large as we please.

In the above theorems a may be a constant or a variable; the only restriction on the value of a is that it shall not become either infinite or zero.

364. It appears from § 157 that one root of the quadratic equation $ax^2+bx+c=0$ is infinite when a is infinitesimal; and that both roots are infinite when a and b are both infinitesimal.

365. Abbreviated Notation. An infinite is often represented by ∞. In § 363, III. and IV. are sometimes written:

$$\frac{a}{0}=\infty, \quad \frac{a}{\infty}=0.$$

The expression $\frac{a}{0}$ cannot be interpreted literally, since we cannot divide by 0; neither can $\frac{a}{\infty} = 0$ be interpreted literally, since we can find no number such that the quotient obtained by dividing a by that number is zero.

$\frac{a}{0} = \infty$ is simply an abbreviated way of writing: *if* $\frac{a}{x} = X$, *and* x *approaches* 0 *as a limit*, X *increases without limit.* $\frac{a}{\infty} = 0$ is simply an abbreviated way of writing: *if* $\frac{a}{X} = x$, *and* X *increases without limit*, x *approaches* 0 *as a limit.*

366. Approach to a Limit. When a variable approaches a limit, it may approach its limit in one of three ways:

(1) The variable may be always less than its limit.

(2) The variable may be always greater than its limit.

(3) The variable may be sometimes less and sometimes greater than its limit.

If x represent the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, x is always less than its limit 2.

If x represent the sum of n terms of the series $3 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots$, x is always greater than its limit 2.

If x represent the sum of n terms of the series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$, we have (§ 227)

$$x = \frac{3 - 3(-\frac{1}{2})^n}{1 + \frac{1}{2}} = 2 - 2(-\tfrac{1}{2})^n.$$

As n is indefinitely increased, x evidently approaches 2 as a limit.

If n is even, x is less than 2; if n is odd, x is greater than 2. Hence, if n be increased by taking each time one more term, x will be alternately less than and greater than 2. If, for example,

$$n = 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7,$$

$$x = 1\tfrac{1}{2}, \quad 2\tfrac{1}{4}, \quad 1\tfrac{7}{8}, \quad 2\tfrac{1}{16}, \quad 1\tfrac{31}{32}, \quad 2\tfrac{63}{64}.$$

In whatever way a variable approaches its limit, the test of § 359 always applies.

367. Equal Variables. *If two variables are equal and are so related that a change in the one produces such a change in the other that they continue equal, and each approaches a limit, then their limits are equal.*

Let x and y be the variables, a and b their respective limits. To prove $a=b$. We have (§ 359)

$$a=x+x', \quad b=y+y',$$

where x' and y' are variables which approach 0 as a limit.

Then, since the equation $x=y$ always holds, we have, by subtraction, $\qquad a-b=x'-y'.$

$x'-y'$ can be made less than any assigned value since x' and y' can each be made less than any assigned value.

Since $x'-y'$ is always equal to the constant $a-b$, $x'-y'$ must be a constant. But the only constant which is less than any assigned value is 0. Therefore $x'-y'=0$, and hence $a-b=0$. $\therefore a=b$.

368. Limit of a Sum. *The limit of the algebraic sum of any finite number of variables is the algebraic sum of their respective limits.*

Let $x, y, z, \ldots\ldots$, be variables;

$a, b, c, \ldots\ldots$, their respective limits.

Then $a-x$, $b-y$, $c-z$, ..., are variables which can each be made less than any assigned value (§ 359).

Then $(a-x)+(b-y)+(c-z)+\cdots\cdots$ can be made less than any assigned value.

For, let v be the numerically greatest of the variables $a-x$, $b-y$, $c-z$, ..., and n the number of variables.

Then, $(a-x)+(b-y)+(c-z)+\cdots\cdots < v+v+v \ldots\ldots$ to n terms

$$< nv;$$

but nv can be made less than any assigned value since n is finite and v can be made less than any assigned value (§ 363, I.).

Therefore, $(a-x)+(b-y)+(c-z)\cdots\cdots$, which is less than nv, can be made less than any assigned value.

$\therefore\ (a+b+c+\cdots\cdots)-(x+y+z+\cdots\cdots)$ can be made less than any assigned value.

$\therefore\ a+b+c+\cdots\cdots$ is the limit of $(x+y+z+\cdots\cdots)$. § 359

369. Limit of a Product. *The limit of the product of two or more variables is the product of their respective limits.*

Let x and y be variables, a and b their respective limits.

To prove that ab is the limit of xy.

Put $x=a-x'$, $y=b-y'$; then x' and y' are variables which can be made less than any assigned value (§ 359).

Now,
$$xy=(a-x')(b-y')$$
$$=ab-ay'-bx'+x'y'.$$
$$\therefore\ ab-xy=ay'+bx'-x'y'.$$

Since every term on the right contains x' or y', the whole right member can be made less than any assigned value (§ 363, I.). Hence, $ab-xy$ can be made less than any assigned value.

$\therefore\ ab$ is the limit of xy (§ 359).

Similarly for three or more variables.

370. Limit of a Quotient. *The limit of the quotient of two variables is the quotient of their limits.*

Let x and y be variables, a and b their respective limits.

Put $a-x=x'$, and $b-y=y'$; then x' and y' are variables with limit 0 (§ 359).

We have $$x=a-x',\ y=b-y',\ \text{and } \frac{x}{y}=\frac{a-x'}{b-y'}.$$

Now $$\frac{a}{b}-\frac{x}{y}=\frac{a}{b}-\frac{a-x'}{b-y'}=\frac{bx'-ay'}{b(b-y')}.$$

The numerator of the last expression approaches 0 as a limit, and the denominator approaches b^2; hence, the expression approaches 0 as a limit (§ 363, I.).

$\therefore\ \frac{a}{b}-\frac{x}{y}$ approaches 0 as a limit. $\therefore\ \frac{a}{b}$ is the limit of $\frac{x}{y}$.

371. Vanishing Fractions. When one or more variables are involved in both numerator and denominator of a fraction, it may happen that for certain values of the variables both numerator and denominator of the fraction vanish. The fraction then assumes the form $\frac{0}{0}$, which is a form without meaning; as even the interpretation of § 365 fails, since the numerator is 0. If, however, there is but *one* variable involved, we may obtain a value as follows:

Let x be the variable, and a the value of x for which the fraction assumes the form $\frac{0}{0}$. Give to x a value a little greater than a, as $a+z$; the fraction will now have a definite value. Find the limit of this last value as z is indefinitely decreased. This limit is called the **limiting value** of the fraction.

(1) Find the limiting value of $\frac{x^2-a^2}{x-a}$ as x approaches a.

When x has the value a, the fraction assumes the form $\frac{0}{0}$.

Put $x=a+z$; the fraction becomes

$$\frac{(a+z)^2-a^2}{(a+z)-a}=\frac{2az+z^2}{z}.$$

Since z is not 0, we can divide by z and obtain $2a+z$.

As z is indefinitely decreased, this approaches $2a$ as a limit. Hence $2a$ is the answer required.

(2) Find the limiting value of $\frac{2x^3-4x+5}{3x^3+2x-1}$ when x becomes infinite.

We have $$\frac{2x^3-4x+5}{3x^3+2x^2-1}=\frac{2-\frac{4}{x^2}+\frac{5}{x^3}}{3+\frac{2}{x}-\frac{1}{x^3}}.$$

As x increases indefinitely, $\frac{1}{x}$ approaches 0 (§ 363, IV.), and the fraction approaches $\frac{2}{3}$. *Ans.*

Exercise 55.

Find the limiting values of:

1. $\dfrac{(4x^2-3)(1-2x)}{7x^3-6x+4}$ when x becomes infinitesimal.

2. $\dfrac{(x^2-5)(x^2+7)}{x^4+35}$ when x becomes infinite.

3. $\dfrac{(x+2)^3}{x^2+4}$ when x becomes infinitesimal.

4. $\dfrac{x^2-8x+15}{x^2-7x+12}$ when x approaches 3.

5. $\dfrac{x^2-9}{x^2+9x+18}$ when x approaches -3.

6. $\dfrac{x(x^2+4x+3)}{x^3+3x^2+5x+3}$ when x approaches -1.

7. $\dfrac{x^3+x^2-2}{x^3+2x^2-2x-1}$ when x approaches 1.

8. $\dfrac{4x+\sqrt{x-1}}{2x-\sqrt{x+1}}$ when x approaches 1.

9. $\dfrac{x-1}{\sqrt{x^2-1}+\sqrt{x-1}}$ when x approaches 1.

10. $\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$ when x approaches 2.

11. $\dfrac{\sqrt{x-a}+\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}$ when x approaches a.

12. If x approaches a as a limit, and n is a positive integer, show that the limit of x^n is a^n.

13. If x approaches a as a limit, and a is not 0, show that the limit of x^n is a^n, where n is a negative integer.

CHAPTER XXVII.

SERIES.

372. Convergency of Series. For an infinite series to be convergent (§ 252) it is *necessary* and *sufficient* that the sum of all the terms after the *nth*, as n is indefinitely increased, should approach 0 as a limit.

Although each of the terms after the *nth* may approach 0 as a limit, their sum may not approach 0 as a limit.

Thus, take the harmonical series,

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \cdots\cdots\ \frac{1}{n},\ \frac{1}{n+1},\ \frac{1}{n+2},\ \cdots\cdots$$

Each term after the *nth* approaches 0 as n increases.

The sum of n terms after the *nth* term is

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n},$$

which is $>\frac{1}{2n}+\frac{1}{2n}+\cdots\cdots$ to n terms; therefore $>n\times\frac{1}{2n}$; that is, $>\frac{1}{2}$.

Now, the first term is 1, the second term is $\frac{1}{2}$, the sum of the next two terms is greater than $\frac{1}{2}$, the sum of the succeeding four terms is greater than $\frac{1}{2}$; and so on. So that, by increasing n indefinitely, the sum will become greater than any finite multiple of $\frac{1}{2}$.

Therefore, the series is *divergent*.

Ex. To determine whether the following series is convergent (§ 267).

$$1+\frac{1}{1}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots\frac{1}{\underline{|n-1}}+\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\cdots\cdots$$

The *nth* term is $\frac{1}{\underline{|n-1}}$. The sum of the remaining terms is

$$\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\frac{1}{\underline{|n+2}}+\cdots\cdots=\frac{1}{\underline{|n}}\left(1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots\cdots\right).$$

This is $< \frac{1}{\lfloor n}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots\cdots\right)$; therefore

$$< \frac{1}{\lfloor n}\left(\frac{1}{1 - \frac{1}{n}}\right), \text{ or } \frac{1}{\lfloor n}\left(\frac{n}{n-1}\right) \text{ (§ 231); that is, } < \frac{1}{(n-1)\lfloor n-1}.$$

But as n increases indefinitely, this last approaches 0 as a limit. Hence, the series is *convergent.*

373. Test for Convergency of a Series. *If the terms of an infinite series are all positive, and the limit of the nth term is* 0, *then if the limit of the ratio of the* $(n+1)$*th term to the nth term, as n is indefinitely increased, is less than* 1, *the series is convergent.*

Let $u_1, u_2, u_3, \ldots\ldots u_n, u_{n+1}, u_{n+2}, \ldots\ldots$ be an infinite series.

Let r represent the limit of the ratio $\frac{u_{n+1}}{u_n}$ as n increases indefinitely, and suppose r to be positive and less than 1.

Let k be some *fixed* number between r and 1, and take k so near 1 that $\frac{u_{n+1}}{u_n}, \frac{u_{n+2}}{u_{n+1}}, \ldots\ldots$, shall each be $< k$.

$$\text{Then, } \frac{u_{n+1}}{u_n} < k, \quad \frac{u_{n+2}}{u_{n+1}} < k, \quad \frac{u_{n+3}}{u_{n+2}} < k, \ \ldots\ldots$$

$$\therefore u_{n+1} < ku_n, \quad u_{n+2} < ku_{n+1}, \quad u_{n+3} < ku_{n+2}, \ \ldots\ldots$$

$$\therefore u_{n+1} < ku_n, \quad u_{n+2} < k^2u_n, \quad u_{n+3} < k^3u_n, \ \ldots\ldots$$

$$\therefore u_{n+1} + u_{n+2} + u_{n+3} + \cdots\cdots < u_n(k + k^2 + k^3 + \cdots\cdots)$$

$$\therefore u_{n+1} + u_{n+2} + u_{n+3} + \cdots\cdots < u_n\frac{k}{1-k}. \qquad \text{(§ 251)}$$

But, by hypothesis, u_n approaches 0 as a limit as n is indefinitely increased. Hence, the series is *convergent.*

Similarly, when r is negative, and between 0 and -1.

Thus, in the series

$$1+\frac{1}{1}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots\frac{1}{\underline{|n-1}}+\frac{1}{\underline{|n}}+\cdots\cdots,$$

$\frac{u_{n+1}}{u_n}=\frac{1}{n}$, and this approaches 0 as a limit as n is indefinitely increased; moreover, the nth term, $\frac{1}{\underline{|n-1}}$, approaches 0 as a limit.

Hence, the series is convergent.

If $r>1$, there must be in the series some term from which the succeeding term is greater than the next preceding term; so that the remaining terms will form an increasing series, and therefore the series is not convergent.

If $r=\pm 1$, this value gives no information as to whether the series is convergent or not; and in such cases other tests must be applied.

If $r<1$, but approaches 1, or -1, as a limit, then no fixed value k can be found which will always lie between r and ± 1, and other tests of convergency must be applied.

Thus, in the infinite series

$$\frac{1}{1^m}+\frac{1}{2^m}+\frac{1}{3^m}+\cdots\cdots+\frac{1}{n^m}+\frac{1}{(n+1)^m}+\cdots\cdots,$$

r, the ratio of the $(n+1)th$ term to the nth term, is

$$\left(\frac{n}{n+1}\right)^m=\left(1-\frac{1}{n+1}\right)^m,$$

which approaches 1 as a limit as n increases.

Suppose m positive and greater than 1; then the first term of the series is 1. The sum of the next two terms is less than $\frac{2}{2^m}$. The sum of the next four terms is less than $\frac{4}{4^m}$. The sum of the next eight terms is less than $\frac{8}{8^m}$; and so on. Hence, the sum of the series is less than $1+\frac{2}{2^m}+\frac{4}{4^m}+\frac{8}{8^m}+\cdots\cdots$, or $<1+\frac{1}{2^{m-1}}+\frac{1}{4^{m-1}}+\frac{1}{8^{m-1}}+\cdots\cdots$, which is evidently convergent when m is positive and greater than 1.

If m is positive and equal to 1, the given series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots,$$

which is the harmonical series shown in § 372 to be divergent.

If m is negative, or less than 1, each term of the series is then greater than the corresponding term in the harmonical series, and hence the series is divergent.

374. Special Case. *If the terms of an infinite series are alternately positive and negative; if, also, the terms continually decrease, and the limit of the nth term is zero, then the series is convergent.*

Consider the infinite series,

$$u_1 - u_2 + u_3 - u_4 + \cdots \mp u_n \pm u_{n+1} \mp u_{n+2} \pm \cdots$$

The sum of the terms after the *nth* term is

$$\pm\left[u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \cdots\right],$$

which may be written

$$\pm\left[u_{n+1} - u_{n+2} + (u_{n+3} - u_{n+4}) + (u_{n+5} - u_{n+6}) + \cdots\right].$$

Since the terms are continually diminishing, each of the groups in either form of expression is positive, and therefore the absolute value of the required sum is seen, from the first form of expression, to be less than u_{n+1}; and from the second form of expression, to be greater than $u_{n+1} - u_{n+2}$. But both u_{n+1} and u_{n+2} approach zero as n increases indefinitely; therefore the sum of the series after the *nth* term approaches zero, and the series is convergent.

In finding the sum of an infinite decreasing series of which the terms are alternately positive and negative, if we stop at any term, the error will be less than the next succeeding term.

The series $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \pm \frac{1}{n} \mp \frac{1}{n+1} \pm \cdots$ is convergent.

For, we may write the series

$$1-\tfrac{1}{2}+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{5}-\tfrac{1}{6})+\cdots\cdots, \text{ or } 1-(\tfrac{1}{2}-\tfrac{1}{3})-(\tfrac{1}{4}-\tfrac{1}{5})-\cdots\cdots,$$

which shows that its sum is greater than $\frac{1}{2}$, and less than 1.

Observe that the present test applies to series in which $\frac{u_{n+1}}{u_n}$ approaches 1, or -1, as a limit; to these series the test of § 373 will not apply.

375. Convergency of the Binomial Series. In the expansion of $(1+x)^n$, the ratio of the $(r+1)th$ term to the rth term is (§ 247)

$$\frac{n-r+1}{r}x, \text{ or } \left(\frac{n+1}{r}-1\right)x.$$

If x is positive, and r greater than $n+1$, $\frac{n+1}{r}-1$ is negative; hence the terms in which r is greater than $n+1$ are alternately positive and negative.

If x is negative, the terms in which r is greater than $n+1$ are all positive. In either case we have

$$\frac{u_{r+1}}{u_r}=\left(\frac{n+1}{r}-1\right)x;$$

as r is indefinitely increased, this approaches the limit $-x$. Hence (§ 373), the series is convergent if x is numerically less than 1.

If n is fractional or negative, the expansion of $(a+b)^n$ must be in the form $a^n\left(1+\frac{b}{a}\right)^n$ if $a>b$; and in the form $b^n\left(1+\frac{a}{b}\right)^n$ if $b>a$ (§ 259).

376. Examples.

(1) For what values of x is the infinite series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\cdots\pm\frac{x^n}{n}\mp\cdots\cdots \text{ convergent?}$$

Here, $$r = \frac{u_{n+1}}{u_n} = \left(\frac{n}{n+1}\right)x = \left(1 - \frac{1}{n+1}\right)x.$$

As n is indefinitely increased, r approaches x as a limit. Hence, the series is convergent when x is numerically less than 1; and divergent when x is numerically greater than 1.

When $x = 1$, the series is convergent by § 374.

When $x = -1$, the series becomes

$$-\left(1 + \frac{1}{2} + \frac{1}{3} \cdots\cdots + \frac{1}{n} + \cdots\cdots\right),$$

the harmonical series already shown to be divergent (§ 372).

(2) For what values of x is the infinite series

$$\frac{x}{1 \times 2} + \frac{x^2}{2 \times 3} + \frac{x^3}{3 \times 4} + \cdots\cdots \frac{x^n}{n(n+1)} \text{ convergent?}$$

Here, $$r = \frac{u_{n+1}}{u_n} = \left(\frac{n}{n+2}\right)x = \left(\frac{1}{1 + \frac{2}{n}}\right)x.$$

As n is indefinitely increased, r approaches x as a limit.

If x is numerically less than 1, the series is convergent.

If x is numerically greater than 1, the series is divergent.

If $x = 1$, every term of the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots\cdots$$

is less than the corresponding term of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$$

This last series is a special case of the series

$$\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \cdots\cdots$$

and is therefore convergent (§ 373).

Hence, $$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots\cdots \text{ is convergent.}$$

If $x = -1$, the series becomes

$$-\frac{1}{1 \times 2} + \frac{1}{2 \times 3} - \frac{1}{3 \times 4} \cdots\cdots$$

and is convergent by § 374.

Exercise 56.

Determine whether the following infinite series are convergent or divergent:

1. $1+\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\dots\dots$

4. $\frac{2}{1^2}+\frac{3}{2^2}+\frac{4}{3^2}+\frac{5}{4^2}+\dots\dots$

2. $1+\frac{1^2}{\underline{|2}}+\frac{2^3}{\underline{|3}}+\frac{3^4}{\underline{|4}}+\dots\dots$

5. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots\dots$

3. $1+\frac{2^2}{\underline{|2}}+\frac{3^3}{\underline{|3}}+\frac{4^4}{\underline{|4}}+\dots\dots$

6. $\frac{1}{1^m}+\frac{1^m}{2^m}+\frac{2^m}{3^m}+\frac{3^m}{4^m}+\dots\dots$

SERIES OF DIFFERENCES.

377. Definitions. If, in any series, we subtract from each term the preceding term, we obtain a first series of differences; in like manner from this last series we may obtain a second series of differences; and so on. In an arithmetical series the second differences all vanish.

There are series, allied to arithmetical series, in which not the first, but the second, or third, etc., differences vanish.

Thus take the series

	1	5	12	24	43	71	110	.....
1st differences,	4	7	12	19	28	39	.....	
2d differences,	3	5	7	9	11	.....		
3d differences,	2	2	2	2	.....			
4th differences,	0	0	0	.....				

In general, if a_1, a_2, a_3, ... be such a series, we have

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	.....
1st differences,	b_1	b_2	b_3	b_4	b_5	b_6	.....	
2d differences,	c_1	c_2	c_3	c_4	c_5	.....		
3d differences,	d_1	d_2	d_3	d_4	.....			
4th differences,	e_1	e_2	e_3	.....				

and finally arrive at differences which all vanish.

378. Any Required Term. For simplicity let us take a series in which the fifth series of differences vanishes. Any other case can be treated in a manner precisely similar. From the manner in which the successive series are formed we shall have:

$$\begin{array}{ll}
a_2 = a_1 + b_1 & a_3 = a_2 + b_2 = a_1 + 2b_1 + c_1 \\
b_2 = b_1 + c_1 & b_3 = b_2 + c_2 = b_1 + 2c_1 + d_1 \\
c_2 = c_1 + d_1 & c_3 = c_2 + d_2 = c_1 + 2d_1 + e_1 \\
d_2 = d_1 + e_1 & d_3 = d_2 + e_2 = d_1 + 2e_1 \\
e_2 = e_1 & e_3 = e_2 \qquad\ = e_1
\end{array}$$

$$\begin{aligned}
a_4 &= a_3 + b_3 = a_1 + 3b_1 + 3c_1 + d_1 \\
b_4 &= b_3 + c_3 = b_1 + 3c_1 + 3d_1 + e_1 \\
c_4 &= c_3 + d_3 = c_1 + 3d_1 + 3e_1 \\
d_4 &= d_3 + e_3 = d_1 + 3e_1
\end{aligned}$$

$$\begin{aligned}
a_5 &= a_4 + b_4 = a_1 + 4b_1 + 6c_1 + 4d_1 + e_1 \\
b_5 &= b_4 + c_4 = b_1 + 4c_1 + 6d_1 + 4e_1 \\
c_5 &= c_4 + d_4 = c_1 + 4d_1 + 6e_1
\end{aligned}$$

$$\begin{aligned}
a_6 &= a_5 + b_5 = a_1 + 5b_1 + 10c_1 + 10d_1 + 5e_1 \\
b_6 &= b_5 + c_5 = b_1 + 5c_1 + 10d_1 + 10e_1
\end{aligned}$$

$$a_7 = a_6 + b_6 = a_1 + 6b_1 + 15c_1 + 20d_1 + 15e_1$$

and so on.

The student will observe that the coefficients in the expression for a_5 are those of the expansion of $(x+y)^4$, and similarly for a_6 and a_7; hence, in general, if we represent a_1, b_1, c_1, etc., by a, b, c, etc., we shall have, putting for the $(n+1)$th term a_{n+1}, the formula

$$a_{n+1} = a + nb + \frac{n(n-1)}{1 \times 2}c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}d + \cdots\cdots$$

Ex. Find the 11th term of 1, 5, 12, 24, 43, 71, 110, ...

Here (§ 377) $a=1,\ b=4,\ c=3,\ d=2,\ e=0$; and $n=10$.

$$\therefore\ a_{11}=a+10b+45c+120d$$
$$=1+40+135+240=416.\ \textit{Ans.}$$

379. Sum of the Series. Form a new series of which the first term is 0, and the first series of differences $a_1, a_2, a_3, \ldots..$ This series will be the following:

$$0,\ a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ a_1+a_2+a_3+a_4,\ \ldots..$$

The $(n+1)$th term of this series will be the sum of n terms of the series $a_1, a_2, a_3, \ldots..$

Find the sum of 11 terms of the series 1, 5, 12, 24, 43, 71, ...

The new series is	0		1		6		18		42		85		156
First differences,		1		5		12		24		43		71	
Second differences,			4		7		12		19		28		
Third differences,				3		5		7		9			
Fourth differences,					2		2		2				

Here $a=0,\ b=1,\ c=4,\ d=3,\ e=2$; and $n=11$.

$$\therefore\ s=a+11b+55c+165d+330e$$
$$=11+220+495+660$$
$$=1386.$$

If s is the sum of n terms of the series $a_1, a_2, a_3, \ldots..$

$$s=0+na+\frac{n(n-1)}{1\times 2}b+\frac{n(n-1)(n-2)}{1\times 2\times 3}c+\ldots..$$

Ex. Find the sum of the squares of the first n natural numbers, $1^2, 2^2, 3^2, 4^2, \ldots..\ n^2$.

Given series,	1		4		9		16		25	n^2
First differences,		3		5		7		9	.....	
Second differences,			2		2		2		.....	
Third differences,				0		0		.....		

Therefore, $a=1,\ b=3,\ c=2,\ d=0$.

These values substituted in the general formula give

$$s=n+\frac{n(n-1)}{1\times 2}\times 3+\frac{n(n-1)(n-2)}{1\times 2\times 3}\times 2$$
$$=n\left\{1+\frac{3n}{2}-\frac{3}{2}+\frac{1}{3}(n^2-3n+2)\right\}$$

$$=\frac{n}{6}\{6+9n-9+2n^2-6n+4\}$$

$$=\frac{n}{6}\{2n^2+3n+1\}=\frac{n(n+1)(2n+1)}{6}.$$

380. Piles of Spherical Shot. I. When the pile is in the form of a triangular pyramid, the summit consists of a single shot resting on three below; and these three rest on a course of six; and these six on a course of ten, and so on, so that the courses will form the series,

$$1,\ 1+2,\ 1+2+3,\ 1+2+3+4, \ldots\ldots, 1+2+\ldots\ldots+n.$$

Given series,	1		3		6		10		15		.....
First differences,		2		3		4		5		.....	
Second differences,			1		1		1		.....		
Third differences,				0		0		.....			

Here, $a=1,\ b=2,\ c=1,\ d=0.$

These values substituted in the general formula give

$$s=n+\frac{n(n-1)}{2}\times 2+\frac{n(n-1)(n-2)}{2\times 3}$$

$$=\frac{n(n+1)(n+2)}{1\times 2\times 3},$$

in which n is the number of balls in the side of the bottom course, or the number of courses.

II. When the pile is in the form of a pyramid with a square base, the summit consists of one shot, the next course consists of four balls, the next of nine, and so on. The number of shot, therefore, is the sum of the series,

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \ldots\ldots\ n^2.$$

Which, by § 379, is

$$\frac{n(n+1)(2n+1)}{1\times 2\times 3},$$

in which n is the number of balls in the side of the bottom course, or the number of courses.

III. When the pile has a base which is rectangular, but not square, the pile will terminate with a single row. Suppose p the number of shot in this row; then the second course will consist of $2(p+1)$ shot; the third course of $3(p+2)$; and the *nth* course of $n(p+n-1)$. Hence the series will be

$$p,\ 2p+2,\ 3p+6,\ \ldots\ldots,\ n(p+n-1).$$

Given series,	p	$2p+2$	$3p+6$	$4p+12$	.....
First differences,	$p+2$	$p+4$	$p+6$	.....	
Second differences,	2	2	.....		
Third differences,	0	.....			

Here, $a=p$, $b=p+2$, $c=2$, $d=0$.

These values substituted in the general formula give

$$s=np+\frac{n(n-1)}{2}(p+2)+\frac{n(n-1)(n-2)}{1\times2\times3}\times 2.$$

$$=\frac{n}{6}\{6p+3(n-1)(p+2)+2(n-1)(n-2)\}$$

$$=\frac{n}{6}(6p+3np-3p+6n-6+2n^2-6n+4)$$

$$=\frac{n}{6}(3np+3p+2n^2-2)$$

$$=\frac{n}{6}(n+1)(3p+2n-2).$$

If n' denote the number in the longest row, then $n'=p+n-1$, and therefore $p=n'-n+1$; and the formula may be written

$$s=\frac{n}{6}(n+1)(3n'-n+1),$$

in which n denotes the number in the width, and n' in the length, of the bottom course.

When the pile is incomplete, compute the number in the pile as if complete, then the number in that part of the pile which is lacking, and take the difference of the results.

Exercise 57.

1. Find the fiftieth term of 1, 3, 8, 20, 43, ...

2. Find the sum of the series 4, 12, 29, 55, ... to 20 terms.

3. Find the twelfth term of 4, 11, 28, 55, 92, ...

4. Find the sum of the series 43, 27, 14, 4, -3, ... to 12 terms.

5. Find the seventh term of 1, 1.235, 1.471, 1.708, ...

6. Find the sum of the series 70, 66, 62.3, 58.9, ... to 15 terms.

7. Find the eleventh term of 343, 337, 326, 310, ...

8. Find the sum of the series 7×13, 6×11, 5×9, ... to 9 terms.

9. Find the sum of n terms of the series 3×8, 6×11, 9×14, 12×17, ...

10. Find the sum of n terms of the series 1, 6, 15, 28, 45, ...

11. Show that the sum of the cubes of the first n natural numbers is the square of the sum of the numbers.

12. Determine the number of shot in the side of the base of a triangular pile which contains 286 shot.

13. The number of shot in the upper course of a square pile is 169, and in the lowest course 1089. How many shot are there in the pile?

14. Find the number of shot in a rectangular pile having 17 shot in one side of the base and 42 in the other.

15. Find the number of shot in the five lower courses of a triangular pile which has 15 in one side of the base.

16. The number of shot in a triangular pile is to the number in a square pile, of the same number of courses, as 22 : 41. Find the number of shot in each pile.

17. Find the number of shot required to complete a rectangular pile having 15 and 6 shot, respectively, in the sides of its upper course.

18. How many shot must there be in the lowest course of a triangular pile that 10 courses of the pile, beginning at the base, may contain 37,020 shot?

19. Find the number of shot in a complete rectangular pile of 15 courses which has 20 shot in the longest side of its base.

20. Find the number of shot in the bottom row of a square pile which contains 2600 more shot than a triangular pile of the same number of courses.

21. Find the number of shot in a complete square pile in which the number of shot in the base and the number in the fifth course above differ by 225.

22. Find the number of shot in a rectangular pile which has 600 in the lowest course and 11 in the top row.

INTERPOLATION.

381. As the expansion of $(a+b)^n$ by the binomial theorem has the same form for fractional as for integral values of n, the formula

$$a_{n+1} = a + nb + \frac{n(n-1)}{1 \times 2}c + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}d + \cdots\cdots$$

may be extended to cases in which n is a fraction, and be employed to insert or *interpolate* terms in a series between given terms.

(1) The cube roots of 27, 28, 29, 30, are 3, 3.03659, 3.07232, 3.10723. Find the cube root of 27.9.

	3		3.03659		3.07232		3.10723
First differences,		0.03659		0.03573		0.03491	
Second differences,			-0.00086		-0.00082		
Third differences,				0.00004			

These values substituted in the general formula give

$$3+\frac{9}{10}(0.03659)-\frac{9}{10}\left(-\frac{1}{10}\right)\left(\frac{0.00086}{2}\right)+\frac{9}{10}\left(-\frac{1}{10}\right)\left(-\frac{11}{10}\right)\left(\frac{0.00004}{6}\right)$$

$$= 3 + 0.032931 + 0.0000387 + 0.00000066$$

$$= 3.03297. \quad \textit{Ans.}$$

(2) Given, $\log 127 = 2.1038$, $\log 128 = 2.1072$, $\log 129 = 2.1106$. Find $\log 127.37$.

	2.1038		2.1072		2.1106
First differences,		0.0034		0.0034	
Second differences,			0		

The second differences vanish, and the required logarithm will be

$$2.1038 + \tfrac{37}{100} \text{ of } 0.0034$$

$$= 2.1038 + 0.001358$$

$$= 2.1052. \quad \textit{Ans.}$$

For the *Nautical Almanac* the Right Ascension and Declination of the Moon are required for every hour of the year. To calculate these directly from the lunar tables would involve an enormous amount of labor. Accordingly the Right Ascension and Declination are calculated from the lunar tables for each noon and midnight of Greenwich time, and those for the other hours are then calculated by interpolation. In this calculation, the following table is useful:

n	$\frac{n(n-1)}{1\times 2}$	$\frac{n(n-1)(n-2)}{1\times 2\times 3}$	n	$\frac{n(n-1)}{1\times 2}$	$\frac{n(n-1)(n-2)}{1\times 2\times 3}$
$\frac{1}{12}$	-0.0382	$+0.0244$	$\frac{7}{12}$	-0.1215	$+0.0574$
$\frac{2}{12}$	-0.0694	$+0.0424$	$\frac{8}{12}$	-0.1111	$+0.0494$
$\frac{3}{12}$	-0.0937	$+0.0547$	$\frac{9}{12}$	-0.0937	$+0.0391$
$\frac{4}{12}$	-0.1111	$+0.0617$	$\frac{10}{12}$	-0.0694	$+0.0270$
$\frac{5}{12}$	-0.1215	$+0.0641$	$\frac{11}{12}$	-0.0382	$+0.0138$
$\frac{6}{12}$	-0.1250	$+0.0625$			

Exercise 58.

Given the declination of the Moon at the following times. Find the declination at each hour in the afternoon of Dec. 1.

				°	′	″
1890.	Dec. 1.	Noon,	N.	22	46	53
		Midnight,		21	28	49
	Dec. 2.	Noon,		19	57	25
		Midnight,		18	13	57
	Dec. 3.	Noon,		16	19	44
		Midnight,		14	15	59

The following table contains the answers.

HOUR.	DECLINATION.			HOUR.	DECLINATION.		
	°	′	″		°	′	″
I.	22	40	55	VII.	22	3	3
II.	22	34	51	VIII.	21	56	24
III.	22	28	42	IX.	21	49	39
IV.	22	22	26	X.	21	42	48
V.	22	16	4	XI.	21	35	51
VI.	22	9	36	XII.	21	28	49

The following table is more useful in general work than the table given on the preceding page. More extended tables will be found in collections of tables.

n	$\frac{n(n-1)}{1\times 2}$	$\frac{n(n-1)(n-2)}{1\times 2\times 3}$	n	$\frac{n(n-1)}{1\times 2}$	$\frac{n(n-1)(n-2)}{1\times 2\times 3}$
0,1	− 0.0450	+ 0.0285	0.6	− 0.1200	+ 0.0560
0.2	− 0.0800	+ 0.0480	0.7	− 0.1050	+ 0.0455
0.3	− 0.1050	+ 0.0595	0.8	− 0.0800	+ 0.0320
0.4	− 0.1200	+ 0.0640	0.9	− 0.0450	+ 0.0165
0.5	− 0.1250	+ 0.0625			

COMPOUND SERIES.

382. It is evident from the form of certain series that they are the sum or the difference of two other series.

(1) Find the sum of the series

$$\frac{1}{1\times 2},\ \frac{1}{2\times 3},\ \frac{1}{3\times 4},\ \ldots\ldots,\ \frac{1}{n(n+1)}.$$

Each term of this series may evidently be expressed in two parts:

$$\frac{1}{1}-\frac{1}{2},\ \frac{1}{2}-\frac{1}{3},\ \ldots\ldots,\ \frac{1}{n}-\frac{1}{n+1};$$

so that the sum will be

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right),$$

in which the second part of each term, except the last, is cancelled by the first part of the next succeeding term.

Hence, the sum is $1-\dfrac{1}{n+1}$.

As n increases without limit, this sum approaches 1 as a limit.

(2) Find the sum of the series

$-\frac{1}{n+1}$

$$\frac{1}{3\times 5},\ \frac{1}{4\times 6},\ \frac{1}{5\times 7},\ \ldots\ldots,\ \frac{1}{n(n+2)}.$$

Each term may be written,

$$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\ \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right),\ \ldots\ldots,\ \frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

$$\therefore\ \text{Sum}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots+\frac{1}{n}-\frac{1}{5}-\frac{1}{6}-\ldots\ldots-\frac{1}{n}-\frac{1}{n+2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{n+2}\right).$$

Hence, the sum is $\dfrac{7}{24}-\dfrac{1}{2(n+2)}$.

As n increases without limit, this sum approaches $\frac{7}{24}$ as a limit.

$-\frac{1}{2(n+1)}$

Exercise 59.

Write down the general term, and sum to n terms, and to an infinite number of terms, the following series:

1. $\frac{1}{1\times 4}+\frac{1}{2\times 5}+\frac{1}{3\times 6}+\dots\dots$

2. $\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\dots\dots$

3. $\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\dots\dots$

4. $\frac{6}{2\times 7}+\frac{6}{7\times 12}+\frac{6}{12\times 17}+\dots\dots$

5. $\frac{1}{5\times 11}+\frac{1}{8\times 14}+\frac{1}{11\times 17}+\dots\dots$

6. $\frac{1}{3\times 8}+\frac{1}{6\times 12}+\frac{1}{9\times 16}+\dots\dots$

7. The series of which the general term is $\frac{1}{n(n+3)}$.

8. The series of which the general term is $\frac{1}{3n(4+4n)}$.

MISCELLANEOUS PROPERTIES OF SERIES.

383. Partial Fractions. To resolve a fraction into *partial fractions* is to express it as the sum of a number of fractions of which the respective denominators are the factors of the denominator of the given fraction. This process is the reverse of the process of *adding* fractions which have different denominators.

Resolution into partial fractions may be easily accomplished by the use of **undetermined coefficients** and the theorem of § 256.

In decomposing a given fraction into its simplest partial fractions, it is important to determine what form the assumed fractions must have.

Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator; moreover, all the factors of the given denominator must be taken as denominators of the assumed fractions.

Since the required partial fractions are to be in their simplest form incapable of further decomposition, the numerator of each required fraction must be assumed with reference to this condition.

Thus, if the denominator is x^n or $(x \pm a)^n$, the assumed fraction must be of the form $\frac{A}{x^n}$ or $\frac{A}{(x \pm a)^n}$; for, if it had the form $\frac{Ax + B}{x^n}$ or $\frac{Ax + B}{(x \pm a)^n}$, it could be decomposed into two fractions, and the partial fractions would not be in the simplest form possible.

When all the monomial factors, and all the binomial factors, of the form $x \pm a$, have been removed from the denominator of the given expression, there may remain quadratic factors which cannot be further resolved; and the numerators corresponding to these quadratic factors may each contain the first power of x, so that the assumed fractions must have either the form $\frac{Ax + B}{x^2 \pm ax + b}$, or the form $\frac{Ax + B}{x^2 + b}$.

(1) Resolve $\frac{3x - 7}{(x - 2)(x - 3)}$ into partial fractions.

The denominators will be $x - 2$ and $x - 3$.

Assume $$\frac{3x - 7}{(x - 2)(x - 3)} \equiv \frac{A}{x - 2} + \frac{B}{x - 3};$$

then $$3x - 7 \equiv A(x - 3) + B(x - 2).$$

$$\therefore A + B = 3 \text{ and } 3A + 2B = 7; \qquad \S\ 256$$

whence, $$A = 1 \text{ and } B = 2.$$

Therefore, $$\frac{3x - 7}{(x - 2)(x - 3)} \equiv \frac{1}{x - 2} + \frac{2}{x - 3}.$$

This identity may be verified by actual multiplication.

(2) Resolve $\dfrac{3}{x^3+1}$ into partial fractions.

Since $x^3+1 \equiv (x+1)(x^2-x+1)$, the denominators will be $x+1$ and x^2-x+1.

Assume $$\frac{3}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1};$$

then $$3 \equiv A(x^2-x+1) + (Bx+C)(x+1)$$
$$\equiv (A+B)x^2 + (B+C-A)x + (A+C);$$

whence, $3 = A+C,\ B+C-A=0,\ A+B=0;$ § 256

and $A=1,\ B=-1,\ C=2.$

Therefore, $$\frac{3}{x^3+1} \equiv \frac{1}{x+1} - \frac{x-2}{x^2-x+1}.$$

(3) Resolve $\dfrac{4x^3-x^2-3x-2}{x^2(x+1)^2}$ into partial fractions.

The denominators may be x, x^2, $x+1$, $(x+1)^2$.

Assume $$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

$$\therefore\ 4x^3-x^2-3x-2 \equiv Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$
$$\equiv (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B;$$

whence, $A+C=4,$ § 256

$2A+B+C+D=-1,$

$A+2B=-3,$

$B=-2;$

or, $A=1,\quad B=-2,\quad C=3,\quad D=-4.$

Therefore, $$\frac{4x^3-x^2-3x-2}{x^2(x+1)^2} \equiv \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x+1} - \frac{4}{(x+1)^2}.$$

Exercise 60.

Resolve into partial fractions:

1. $\dfrac{7x+1}{(x+4)(x-5)}$.
2. $\dfrac{6}{(x+3)(x+4)}$.
3. $\dfrac{5x-1}{(2x-1)(x-5)}$.
4. $\dfrac{x-2}{x^2-3x-10}$.
5. $\dfrac{3}{x^3-1}$.
6. $\dfrac{x^2-x-3}{x(x^2-4)}$.

7. $\dfrac{3x^2-4}{x^2(x+5)}$.

8. $\dfrac{7x^2-x}{(x-1)^2(x+2)}$.

9. $\dfrac{2x^2-7x+1}{x^3-1}$.

10. $\dfrac{7x-1}{6x^2-5x+1}$.

11. $\dfrac{13x+46}{12x^2-11x-15}$.

12. $\dfrac{2x^2-11x+5}{x^3-x^2-11x+15}$.

---

384. Expansion in Series. A series which is obtained from a given expression is called the **expansion** of that expression (§ 243). The given expression is called the **generating function** of the series.

Thus (§ 250), the expression $\dfrac{1}{1-x}$ is the generating function of the infinite series $1+x+x^2+x^3+\cdots\cdots$

When the expression is a finite series, the generating function is equal to the expansion for all values of the symbols involved.

Thus, $$\left(\frac{1+2x^2}{x}\right)^3 \equiv \frac{1}{x^3}+\frac{6}{x}+12x+8x^3.$$

When the expansion is an infinite series, the generating function is equal to the expansion for only such values of the symbols involved as make the expansion a convergent series.

Thus, $\dfrac{1}{1-x}$ is equal to the series $1+x+x^2+x^3+\cdots\cdots$ when, and only when, x is numerically less than 1 (§ 251).

385. The expansion of a given expression may be found:

By division,

By the binomial theorem,

By the method of undetermined coefficients,

By other methods, which involve a knowledge of the Differential Calculus.

(1) Expand $\dfrac{x}{1+x^2}$ in ascending powers of x.

Divide x by $1+x^2$, then

$$\frac{x}{1+x^2}=x-x^3+x^5-\dots..$$

provided x is so taken that the series is convergent. By § 373 x must be numerically less than 1.

(2) Expand $\dfrac{x}{1+x^2}$ in descending powers of x.

Divide x by x^2+1, then

$$\frac{x}{1+x^2}=\frac{1}{x}-\frac{1}{x^3}+\frac{1}{x^5}-\dots..$$

provided x is so taken that the series is convergent. By § 373 x must be numerically greater than 1.

In the two preceding examples we have found an expansion of $\dfrac{x}{1+x^2}$ for all values of x except ± 1.

(3) Expand $\dfrac{x}{1+x^2}$ in ascending powers of x by the binomial theorem.

$$\frac{1}{1+x^2}=(1+x^2)^{-1}=1-x^2+x^4-\dots..$$

$$\therefore \frac{x}{1+x^2}=x-x^3+x^5-\dots..$$

provided x is so taken that the series is convergent.

(4) Expand $\dfrac{2+3x}{1+x+x^2}$ in ascending powers of x.

Assume $$\frac{2+3x}{1+x+x^2}=A+Bx+Cx^2+Dx^3\dots..;$$

then, by clearing of fractions,

$$\begin{aligned}2+3x=A+Bx&+Cx^2+Dx^3+\dots..\\ +Ax&+Bx^2+Cx^3+\dots..\\ &+Ax^2+Bx^3+\dots..\end{aligned}$$

By § 256, $A = 2,\ B + A = 3,\ C + B + A = 0,\ D + C + B = 0;$
whence $B = 1, \qquad C = -3, \qquad D = 2,$ and so on.

$$\therefore\ \frac{2 + 3x}{1 + x + x^2} = 2 + x - 3x^2 + 2x^3 + x^4 - 3x^5 + \cdots\cdots$$

The series is of course equal to the fraction for only such values of x as make the series convergent.

Remark. In employing the method of Undetermined Coefficients, the form of the given expression must determine what powers of the variable x must be assumed. It is necessary and sufficient that the assumed equation, when simplified, shall have in the right member all the powers of x that are found in the left member.

If any powers of x occur in the *right* member that are not in the *left* member, the coefficients of these powers in the right member will vanish, so that in this case the method still applies; but if any powers of x occur in the *left* member that are not in the *right* member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x; and this leads to absurd results. Thus, if it were assumed in problem (4) that

$$\frac{2 + 3x}{1 + x + x^2} = Ax + Bx^2 + Cx^3 + \cdots\cdots,$$

there would be in the simplified equation no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x, 2, which is $2x^0$, would have to be put equal to $0x^0$; that is, $2 = 0$, an absurdity.

(5) Expand $(a - x)^{\frac{1}{2}}$ in a series of ascending powers of x.

Assume $\qquad (a - x)^{\frac{1}{2}} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots$

Square, $a - x = A^2 + 2ABx + (2AC + B^2)x^2 + (2AD + 2BC)x^3 + \cdots\cdots$

Therefore, by § 256,

$$A^2 = a,\ 2AB = -1,\ 2AC + B^2 = 0,\ 2AD + 2BC = 0, \text{ etc.,}$$

and $\quad A = a^{\frac{1}{2}},\ B = -\dfrac{1}{2a^{\frac{1}{2}}},\ C = -\dfrac{1}{8a^{\frac{3}{2}}},\ D = -\dfrac{1}{16a^{\frac{5}{2}}}.$

Hence, $\quad (a - x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \dfrac{x}{2a^{\frac{1}{2}}} - \dfrac{x^2}{8a^{\frac{3}{2}}} - \dfrac{x^3}{16a^{\frac{5}{2}}} - \cdots\cdots$ Cf. § 258

(6) Expand $\dfrac{7+x}{(1+x)(1+x^2)}$ in a series of ascending powers of x.

Assume $$\frac{7+x}{(1+x)(1+x^2)} \equiv \frac{A}{1+x} + \frac{Bx+C}{1+x^2}.$$

$$\therefore\ 7+x \equiv (A+B)x^2 + (B+C)x + (A+C).$$

$$\therefore\ A+B=0,\quad B+C=1,\quad A+C=7.$$

Whence, $\quad A=3,\qquad B=-3,\qquad C=4.$

$$\therefore\ \frac{7+x}{(1+x)(1+x^2)} \equiv \frac{3}{1+x} + \frac{4-3x}{1+x^2}.$$

But $$\frac{3}{1+x} = 3\left(\frac{1}{1+x}\right) = 3(1-x+x^2-x^3+x^4-\cdots\cdots)$$
$$= 3-3x+3x^2-3x^3+3x^4-\cdots\cdots,$$

and $$\frac{4-3x}{1+x^2} = (4-3x)\left(\frac{1}{1+x^2}\right) = (4-3x)(1-x^2+x^4-\cdots\cdots)$$
$$= 4-3x-4x^2+3x^3+4x^4-\cdots\cdots$$

Adding the two series,

$$\frac{7+x}{(1+x)(1+x^2)} = 7-6x-x^2+7x^4\cdots\cdots$$

386. Reversion of a Series. Given

$$y = ax + bx^2 + cx^3 + dx^4 + \cdots\cdots,$$

where the series is convergent, to find x in terms of y.

Assume $\quad x = Ay + By^2 + Cy^3 + Dy^4 + \cdots\cdots$

In this series for y put $ax + bx^2 + cx^3 + dx^4 + \cdots\cdots$; the result is

$$x = aAx + \begin{array}{c|c|c|c} bA & x^2 + cA & x^3 + dA & x^4 + \cdots\cdots \\ + a^2B & +2abB & +b^2B & \\ & +a^3C & +2acB & \\ & & +3a^2bC & \\ & & +a^4D & \end{array}$$

Comparing coefficients (§ 256),

$$aA = 1;\ \ bA + a^2B = 0;\ \ cA + 2abB + a^3C = 0;$$
$$dA + b^2B + 2acB + 3a^2bC + a^4D = 0.$$
$$\therefore\ A = \frac{1}{a},\quad B = -\frac{b}{a^3},\quad C = \frac{2b^2 - ac}{a^5},$$
$$D = -\frac{5b^3 - 5abc + a^2d}{a^7},\ \text{etc.}$$

(1) Given $y = x + x^2 + x^3 + \cdots\cdots$; find x in terms of y.

Here, $a = 1,\ \ b = 1,\ \ c = 1,\ \ d = 1, \ldots\ldots$

$A = 1,\ \ B = -1,\ \ C = 1,\ \ D = -1, \ldots\ldots$

Hence, $x = y - y^2 + y^3 - y^4 + \cdots\cdots$

(2) Revert $y = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots\cdots$.

Here, $a = 1,\ \ b = -\frac{1}{2},\ \ c = \frac{1}{3},\ \ d = -\frac{1}{4}, \ldots\ldots$

$\therefore\ A = 1,\ \ B = \dfrac{1}{\underline{|2}},\ \ C = \dfrac{1}{\underline{|3}},\ \ D = \dfrac{1}{\underline{|4}}, \ldots\ldots$

Hence, $x = y + \dfrac{y^2}{\underline{|2}} + \dfrac{y^3}{\underline{|3}} + \dfrac{y^4}{\underline{|4}} + \cdots\cdots$

Exercise 61.

Expand to four terms in ascending powers of x:

1. $\dfrac{1}{1 - 2x}$.
2. $\dfrac{1}{2 - 3x}$.
3. $\dfrac{1 + x}{2 + 3x}$.
4. $\dfrac{1 - x}{1 + x + x^2}$.
5. $\dfrac{5 - 2x}{1 + x - x^2}$.
6. $\dfrac{4x - 6x^2}{1 - 2x + 3x^2}$.
7. $\dfrac{x(x - 1)}{(x+1)(x^2+1)}$.
8. $\dfrac{x^2 - x + 1}{x^2(x^2 - 1)}$.
9. $\dfrac{2x^2 - 1}{x(x^3 + 1)}$.

Expand to four terms in descending powers of x:

10. $\dfrac{4}{2+x}$. 12. $\dfrac{5-2x}{1+3x-x^2}$. 14. $\dfrac{3x-2}{x(x-1)^2}$.

11. $\dfrac{2-x}{3+x}$. 13. $\dfrac{x^2-x+1}{x(x-2)}$. 15. $\dfrac{x^2-x+1}{(x-1)(x^2+1)}$.

Revert:

16. $y = x - 2x^2 + 3x^3 - 4x^4 + \cdots\cdots$

17. $y = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots\cdots$

18. $y = x + \dfrac{x^2}{1\cdot 2} + \dfrac{x^3}{2\cdot 3} + \dfrac{x^4}{3\cdot 4} + \cdots\cdots$

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387. The following series have been already studied:

(1) Arithmetical series (§§ 218–224).
(2) Geometrical series (§§ 225–231).
(3) Harmonical series (§§ 232–235).
(4) Expansions obtained by the binomial theorem (§§ 239–260).
(5) Series of Differences (§§ 377–380).

We shall now consider series obtained by division, or by the method of undetermined coefficients (§ 385).

388. Recurring Series. From the expression $\dfrac{1+x}{1-2x-x^2}$ we obtain by actual division, or by the method of undetermined coefficients, the infinite series

$$1 + 3x + 7x^2 + 17x^3 + 41x^4 + 99x^5 + \cdots\cdots$$

In this series any required term after the second is found by multiplying the term before the required term by $2x$, the term before that by x^2, and adding the products.

Thus, take the fifth term:

$$41x^4 = 2x(17x^3) + x^2(7x^2).$$

In general, if u_n represent the *nth* term,

$$u_n \equiv 2xu_{n-1} + x^2u_{n-2}.$$

A series in which a relation of this character exists is called a **recurring series.** Recurring series are of the *first*, *second*, *third*, ... **order**, according as each term is dependent upon *one*, *two*, *three*, ... preceding terms.

A recurring series of the first order is evidently an ordinary geometrical series.

In an arithmetical, or geometrical, series any required term can be found when the term immediately preceding is given. In a series of differences, or a recurring series, several preceding terms must be given if any required term is to be found.

The relation which exists between the successive terms is called the **identical relation** of the series; the coefficients of this relation, when all the terms are transposed to the left member, is called the **scale of relation** of the series.

Thus, in the series

$$1 + 3x + 7x^2 + 17x^3 + 41x^4 + 99x^5 + \ldots\ldots$$

the identical relation is

$$u_n \equiv 2xu_{n-1} + x^2u_{n-2};$$

and the scale of relation is

$$1 - 2x - x^2.$$

389. If the identical relation of the series is given, any required term can be found when a sufficient number of preceding terms are given.

Conversely, the identical relation can be found when a sufficient number of terms are given.

(1) Find the identical relation of the recurring series

$$1 + 4x + 14x^2 + 49x^3 + 171x^4 + 597x^5 + 2084x^6 + \cdots\cdots$$

Try first a relation of the second order.

Assume $$u_n = pxu_{n-1} + qx^2u_{n-2}.$$

Putting $n = 3$, and, then, $n = 4$,

$$14 = 4p + q,$$
$$49 = 14p + 4q;$$

whence, $$p = \tfrac{7}{2},\ q = 0.$$

This gives a relation which does not hold true for the fifth and following terms.

Try next a relation of the third order.

Assume $$u_n = pxu_{n-1} + qx^2u_{n-2} + rx^3u_{n-3}.$$

Putting $n = 4$, then $n = 5$, then $n = 6$.

$$\begin{aligned} 49 &= 14p + 4q + r, \\ 171 &= 49p + 14q + 4r, \\ 597 &= 171p + 49q + 14r; \end{aligned}$$

whence, $$p = 3,\ q = 2,\ r = -1.$$

This gives the relation

$$u_n \equiv 3xu_{n-1} + 2x^2u_{n-2} - x^3u_{n-3}$$

which is found to hold true for the seventh term.

The *scale of relation* is $1 - 3x - 2x^2 + x^3$.

(2) Find the eighth term of the above series.

Here, $$\begin{aligned} u_8 &\equiv 3xu_7 + 2x^2u_6 - x^3u_5 \\ &\equiv 3x(2084x^6) + 2x^2(597x^5) - x^3(171x^4) \\ &\equiv 7275x^7.\ \textit{Ans.} \end{aligned}$$

390. Sum of an Infinite Series. By the *sum* of an infinite convergent *numerical* series is meant the limit which the sum of n terms of the series approaches as n is indefinitely increased; a *divergent* numerical series has no true sum.

By the sum of an infinite series of which the successive terms involve one or more *variables* is meant the *generating function* of the series (§ 384); that is, *the expression of which the series is the expansion.*

The generating function is a true sum when, and only when, the series is convergent.

The process of finding the generating function is called *summation* of the series.

391. Sum of a Recurring Series. The sum of a recurring series can be found by a method analogous to that by which the sum of a geometrical series is found (§ 227).

Take, for example, a recurring series of the second order in which the identical relation is

$$u_k \equiv pu_{k-1} + qu_{k-2},$$

or

$$u_k - pu_{k-1} - qu_{k-2} \equiv 0.$$

Let s represent the sum of the series; then

$$s = u_1 + u_2 + u_3 + \cdots\cdots \quad u_{n-1} + u_n,$$

$$-ps = \quad -pu_1 - pu_2 - \cdots\cdots - pu_{n-2} - pu_{n-1} - pu_n,$$

$$-qs = \quad -qu_1 - \cdots\cdots - qu_{n-3} - qu_{n-2} - qu_{n-1} - qu_n.$$

Now, by the identical relation,

$$u_3 - pu_2 - qu_1 = 0,\ u_4 - pu_3 - qu_2 = 0, \cdots\cdots u_n - pu_{n-1} - qu_{n-2} = 0.$$

Therefore, adding the above series,

$$s = \frac{u_1 + (u_2 - pu_1)}{1 - p - q} - \frac{pu_n + q(u_n + u_{n-1})}{1 - p - q}.$$

Observe that the denominator is the *scale of relation.*

If the series is infinite and convergent, u_n and u_{n-1} each approaches 0 as a limit, and s approaches as a limit the fraction $\dfrac{u_1 + (u_2 - pu_1)}{1 - p - q}$.

If the series is infinite, whether convergent or not, this fraction is the *generating function* of the series.

For a recurring series of the third order of which the identical relation is

$$u_k \equiv pu_{k-1} + qu_{k-2} + ru_{k-3},$$

we find $$s = \frac{u_1 + (u_2 - pu_1) + (u_3 - pu_2 - qu_1)}{1 - p - q - r}$$
$$- \frac{pu_n + q(u_n + u_{n-1}) + r(u_n + u_{n-1} + u_{n-2})}{1 - p - q - r}.$$

Similarly for any recurring series.

(1) Find the generating function of the infinite recurring series

$$1 + 4x + 13x^2 + 43x^3 + 142x^4 + \cdots\cdots$$

By § 389 the identical relation is found to be

$$u_k \equiv 3xu_{k-1} + x^2u_{k-2}.$$

Hence, $$s = 1 + 4x + 13x^2 + 43x^3 + 142x^4 + \cdots\cdots$$
$$-3xs = \quad -3x - 12x^2 - 39x^3 - 129x^4 - \cdots\cdots$$
$$-x^2s = \quad\quad -x^2 - 4x^3 - 13x^4 - \cdots\cdots$$

Adding, $$(1 - 3x - x^2)s = 1 + x,$$
$$s = \frac{1 + x}{1 - 3x - x^2}.$$

(2) Find the generating function and the general term of the infinite recurring series

$$1 - 7x - x^2 - 43x^3 - 49x^4 - 307x^5 - \cdots\cdots$$

Here $$u_k \equiv xu_{k-1} + 6x^2u_{k-2}.$$

$$s = 1 - 7x - x^2 - 43x^3 - 49x^4 - \cdots\cdots$$
$$-xs = \quad -x + 7x^2 + x^3 + 43x^4 + \cdots\cdots$$
$$-6x^2s = \quad\quad -6x^2 + 42x^3 + 6x^4 + \cdots\cdots$$
$$s = \frac{1 - 8x}{1 - x - 6x^2} = \frac{1 - 8x}{(1 + 2x)(1 - 3x)}.$$

By § 383 we find

$$\frac{1-8x}{(1+2x)(1-3x)}=\frac{2}{1+2x}-\frac{1}{1-3x}.$$

By the binomial theorem or by actual division,

$$\frac{1}{1+2x}=1-2x+2^2x^2-2^3x^3+\dots\dots+2^r(-1)^rx^r+\dots\dots,$$

$$\frac{1}{1-3x}=1+3x+3^2x^2+3^3x^3+\dots\dots 3^rx^r+\dots\dots$$

Hence the general term of the given series is

$$[2^{r+1}(-1)^r-3^r]x^r.$$

(3) Find the identical relation in the series

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+\dots\dots$$

The identical relation is found from the equations

$$16=9p+4q+r,$$
$$25=16p+9q+4r,$$
$$36=25p+16q+9r,$$

to be

$$u_k\equiv 3u_{k-1}-3u_{k-2}+u_{k-3}.$$

Exercise 62.

Find the identical relation and generating function of:

1. $1+2x+7x^2+23x^3+76x^4+\dots\dots$
2. $3+2x+3x^2+7x^3+18x^4+\dots\dots$

Find the generating function and the general term of:

3. $2+3x+5x^2+9x^3+17x^4+33x^5+\dots\dots$
4. $7-6x+9x^2+27x^3+54x^4+189x^5+\dots\dots$
5. $1+5x+9x^2+13x^3+17x^4+21x^5+\dots\dots$
6. $1+x-7x^3+33x^4-130x^5+499x^6+\dots\dots$
7. $3+6x+14x^2+36x^3+98x^4+276x^5+\dots\dots$

Find the sum of n terms of:

8. $2+5+10+17+26+37+50+\dots\dots$
9. $1^3+2^3+3^3+4^3+5^3+\dots\dots$

EXPONENTIAL AND LOGARITHMIC SERIES.

392. Exponential Series. By the binomial theorem

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\times\frac{1}{n}+\frac{nx(nx-1)}{1\times 2}\times\frac{1}{n^2}$$
$$+\frac{nx(nx-1)(nx-2)}{1\times 2\times 3}\times\frac{1}{n^3}+\ldots\ldots$$
$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\ldots\ldots \quad (1)$$

This equation is true for all real values of x, since the binomial theorem may readily be extended to the case of incommensurable exponents by the method of § 264; it is, however, only true for values of n numerically greater than 1, since $\frac{1}{n}$ must be numerically less than 1 (§ 375).

As (1) is true for all values of x, it is true when $x=1$.

$$\therefore\ \left(1+\frac{1}{n}\right)^{n}=1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\ldots\ldots \quad (2)$$

But
$$\left[\left(1+\frac{1}{n}\right)^{n}\right]^{x}=\left(1+\frac{1}{n}\right)^{nx} \qquad \text{§ 264}$$

Hence, from (1) and (2),

$$\left[1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\ldots\ldots\right]^{x}$$
$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\ldots\ldots$$

This last equation is true for all values of n numerically greater than 1. Take the limits of the two members as n increases without limit. Then (§ 367)

$$\left(1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots\right)^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots\cdots, \quad (3)$$

and this is true for all values of x. It is easily seen by § 373 that the second series is convergent for all values of x; the first series was proved convergent in § 372.

The sum of the infinite series in parenthesis is called the natural base (§ 267), and is generally represented by e; hence, by (3),

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots\cdots \qquad \textbf{A}$$

To calculate the value of e we proceed as follows:

$$\begin{array}{r|l}
2 & 1.000000 \\ \hline
3 & 0.500000 \\ \hline
4 & 0.166667 \\ \hline
5 & 0.041667 \\ \hline
6 & 0.008333 \\ \hline
7 & 0.001388 \\ \hline
8 & 0.000198 \\ \hline
9 & 0.000025 \\ \hline
 & 0.000003
\end{array}$$

Adding, $e=2.71828.$

To ten places, $e=2.7182818284.$

393. In **A** put cx in place of x; then

$$e^{cx}=1+cx+\frac{c^2x^2}{\underline{|2}}+\frac{c^3x^3}{\underline{|3}}+\cdots\cdots$$

Put $e^c=a$; then $c=\log_e a$, and $e^{cx}=a^x$.

$$\therefore\ a^x=1+x\log_e a+\frac{x^2(\log_e a)^2}{\underline{|2}}+\frac{x^3(\log_e a)^3}{\underline{|3}}+\cdots\cdots \qquad \textbf{B}$$

The series in **B** is known as the **exponential series**; **B** reduces to **A** when we put e for a.

394. Logarithmic Series. In **A** put $e^x = 1 + y$; then

$$x = \log_e(1+y), \text{ and by } \mathbf{A},$$

$$y = x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots\cdots$$

Revert the series (§ 386), and we obtain

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots$$

But $\qquad x = \log_e(1+y).$

$$\therefore \log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots \qquad \mathbf{C}$$

Similarly from **B**,

$$\log_a(1+y) = \frac{1}{\log_e a}\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\cdots\right). \qquad \mathbf{D}$$

The series in **D** is known as the **logarithmic series**; **D** reduces to **C** when we put e for a.

In **C** and **D** y must be between -1 and $+1$, or be equal to $+1$, in order to have the series convergent (§ 376, Ex. 1).

395. Modulus. Comparing **C** and **D** we obtain

$$\log_a(1+y) = \frac{1}{\log_e a}\log_e(1+y);$$

or, putting N for $1 + y$,

$$\log_a N = \frac{1}{\log_e a}\log_e N.$$

Hence, to change logarithms from the base e to the base a, multiply by $\dfrac{1}{\log_e a} = \log_a e$; and conversely (§ 283).

The number by which *natural logarithms* must be multiplied to obtain logarithms to the base a is called the **modulus** of the system of logarithms of which a is the base.

Thus, the modulus of the common system is $\log_{10} e$ (§ 285).

396. Calculation of Logarithms. Since the series in **C** and **D** are not convergent when x is numerically greater than 1, they are not adapted to the calculation of logarithms in general. We obtain a convenient series as follows:

The equation

$$\log_e(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\cdots\cdots \qquad (1)$$

holds true for all values of y numerically less than 1; therefore, if it holds true for any particular value of y, it will hold true when we put $-y$ for y; this gives

$$\log_e(1-y)=-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\cdots\cdots \qquad (2)$$

Subtracting (2) from (1), since

$$\log_e(1+y)-\log_e(1-y)=\log_e\left(\frac{1+y}{1-y}\right),$$

we find $\log_e\left(\dfrac{1+y}{1-y}\right)=2\left(y+\dfrac{y^3}{3}+\dfrac{y^5}{5}+\cdots\cdots\right).$

Put $y=\dfrac{1}{2z+1}$; then $\dfrac{1+y}{1-y}=\dfrac{z+1}{z}$,

and $\log_e\left(\dfrac{z+1}{z}\right)=\log_e(z+1)-\log_e z$

$$=2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\cdots\cdots\right). \qquad \textbf{E}$$

This series is convergent for all positive values of z.

Logarithms to any base a can be calculated by the corresponding series obtained from **D**; viz.:

$\log_a(z+1)-\log_a z$

$$=\frac{2}{\log_e a}\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\cdots\cdots\right). \qquad \textbf{F}$$

(1) Calculate to six places of decimals $\log_e 2$, $\log_e 3$, $\log_e 10$, $\log_{10} e$.

In **E** put $\quad z = 1$; then $2z+1=3$, $\log_e z = 0$,

and $$\log_e 2 = \frac{2}{3} + \frac{2}{3\times 3^3} + \frac{2}{5\times 3^5} + \frac{2}{7\times 3^7} + \cdots\cdots$$

The work may be arranged as follows:

3	2.0000000		
9	0.6666667	$\div$ 1	= 0.6666667
9	0.0740741	$\div$ 3	= 0.0246914
9	0.0082305	$\div$ 5	= 0.0016461
9	0.0009145	$\div$ 7	= 0.0001306
9	0.0001016	$\div$ 9	= 0.0000113
9	0.0000113	$\div$ 11	= 0.0000010
	0.0000013	$\div$ 13	= 0.0000001
		$\log_e 2$	= 0.693147

$$\log_e 3 = \log_e 2 + \frac{2}{5} + \frac{2}{3\times 5^3} + \frac{2}{5\times 5^5} + \cdots\cdots$$
$$= 1.0986123.$$
$$\log_e 9 = \log_e(3^2) = 2\log_e 3 = 2.1972246.$$
$$\log_e 10 = \log_e 9 + \frac{2}{19} + \frac{2}{3\times 19^3} + \frac{2}{5\times 19^5} + \cdots\cdots$$
$$= 2.1972246 + 0.1053606$$
$$= 2.302585.$$
$$\log_{10} e = \frac{1}{\log_e 10} = 0.434294.$$

Hence, the *modulus* of the common system is 0.434294.

To *ten* places of decimals:

$$\log_e 10 = 2.3025850928,$$
$$\log_{10} e = 0.4342944819.$$

For calculating common logarithms we use the series in **F**.

$$\log_{10}(z+1) - \log_{10} z$$
$$= 0.8685889638\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\cdots\right).$$

(2) Calculate to five places of decimals $\log_{10} 11$.

Put $z = 10$; then $2z + 1 = 21$, $\log z = 1$.

$$\log 11 = 1 + 0.868588\left(\frac{1}{21} + \frac{1}{3 \times 21^3} + \frac{1}{5 \times 21^5} + \cdots\right)$$

$$\begin{array}{r|l}
21 & 0.868588 \\
441 & 0.041361 \div 1 = 0.041361 \\
 & \quad 94 \div 3 = 31 \\
 & \hline \\
 & 0.041392 \\
 & 1 \\
\end{array}$$

$$\log_{10} 11 = 1.04139$$

In calculating logarithms, the accuracy of the work may be tested every time we come to a composite number by adding together the logarithms of the several factors (§ 365). In fact the logarithms of composite numbers may be found by addition, and then only the logarithms of prime numbers need be found by the series.

397. Limit of $\left(1 + \frac{x}{n}\right)^n$. By the binomial theorem,

$$\left(1+\frac{x}{n}\right)^n = 1 + n \times \frac{x}{n} + \frac{n(n-1)}{1 \times 2} \times \frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \times \frac{x^3}{n^3} + \cdots$$

$$= 1 + x + \frac{1 - \frac{1}{n}}{\lfloor 2}x^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}x^3 + \cdots$$

This equation is true for all values of n greater than x (§ 375). Take the limit as n increases without limit, x remaining finite; then

$$\lim_{n \text{ infinite}} \left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

$$= e^x$$

$$= \lim_{n \text{ infinite}} \left(1+\frac{1}{n}\right)^{nx}. \qquad \text{§ 392}$$

Exercise 63.

1. Show that the infinite series

$$\frac{1}{1\times 2}-\frac{1}{2\times 2^2}+\frac{1}{3\times 2^3}-\frac{1}{4\times 2^4}+\dots\dots$$

is convergent, and find its sum.

2. Find the limit which $\sqrt[n]{1+nx}$ approaches as n approaches 0 as a limit.

3. Prove that $\dfrac{1}{e}=2\left(\dfrac{1}{\lfloor 3}+\dfrac{2}{\lfloor 5}+\dfrac{3}{\lfloor 7}+\dots\dots\right)$.

4. Calculate to four places, $\log_e 4$, $\log_e 5$, $\log_e 6$, $\log_e 7$.

5. Find to four places the moduli of the systems of which the bases are: 2, 3, 4, 5, 6, 7.

6. Show that

$$\log_e\left(\frac{8}{e}\right)=\frac{5}{1\times 2\times 3}+\frac{7}{3\times 4\times 5}+\frac{9}{5\times 6\times 7}+\dots\dots$$

7. Show that

$$\log_e a-\log_e b=\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\dots\dots$$

8. Show that, if x is positive,

$$x+\frac{1}{x}-\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)+\frac{1}{3}\left(x^3+\frac{1}{x^3}\right)-\dots\dots=\log_e\left(2+x+\frac{1}{x}\right).$$

9. Show that $1+\dfrac{2^3}{\lfloor 2}+\dfrac{3^3}{\lfloor 3}+\dfrac{4^3}{\lfloor 4}=5e$.

10. Show that $e^{x\sqrt{-1}}=X+Y\sqrt{-1}$ where

$$X=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\dots\dots,\quad Y=x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\frac{x^7}{\lfloor 7}+\dots\dots$$

11. Expand $\dfrac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}$ in ascending powers of x.

12. Expand $\dfrac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}$ in ascending powers of x.

CHAPTER XXVIII.

DETERMINANTS.

398. Origin. Solving the two simultaneous equations

$$a_1x + b_1y = c_1,$$

$$a_2x + b_2y = c_2,$$

we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Similarly, from the three simultaneous equations

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3,$$

we obtain

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1},$$

with similar expressions for y and z.

The numerators and denominators of these fractions are examples of expressions which often occur in algebraic work, and for which it is therefore convenient to have a special name; such expressions are called **determinants.**

399. Definitions. Determinants are usually written in a compact form, called the *square form.*

Thus, $a_1b_2 - a_2b_1$ is written $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$,

and $$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

is written $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This square form is sometimes written in a still more abbreviated form. Thus, the last two determinants are written $|a_1\ b_2|$ and $|a_1\ b_2\ c_3|$. This last notation should, however, always suggest the square form; in any problem it will generally be advisable to write out this abbreviated form in the complete square form.

The individual symbols a_1, a_2, b_1, b_2, ..., are called **elements.**

A horizontal line of elements is called a **row**; a vertical line a **column.**

The two lines a_1, b_2, c_3 and a_3, b_2, c_1 are called **diagonals**; the first the **principal diagonal,** the second the **secondary diagonal.**

The **order** of a determinant is the number of elements in a row or column.

Thus, the last two determinants are of the second and third orders, respectively.

The expression of which the square form is an abbreviation is called the **expanded form,** or simply the **expansion,** of the determinant.

The several terms of the expansion are called **terms** of the determinant.

Thus the expansion of $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is $a_1b_2 - a_2b_1$.

REMARK. By some writers *constituent* is used where we use *element*, and *element* where we use *term*.

400. General Definition. In general, a determinant of the *nth* order is an expression involving n^2 elements arranged in n rows of n elements each; the expansion, that is, the expression for which the square form is an abbreviation, being found as follows:

Form all the possible products of n elements each that can be formed by taking one, *and only one*, element from each row, and one, *and only one*, element from each column; prefix to each of the products thus formed either $+$ or $-$ (which sign is to be determined by a rule to be given in the following sections), and take the sum of all these products.

Nearly all the properties of determinants can be obtained directly from this definition and the rule of signs (§ 403 or § 404). This will be the method followed in the present chapter. It is therefore of the utmost importance that the student should thoroughly understand the present and the four following sections.

401. Inversions of Order. In any particular determinant the letters and subscripts in the principal diagonal are said to be in the *natural order*. If the letters, or subscripts, are taken in any other order, there will be one or more *inversions* of order.

Thus, if 1, 2, 3, 4, 5 be the natural order, in the order 2, 3, 5, 1, 4, there will be four inversions: 2 before 1, 3 before 1, 5 before 1, 5 before 4.

Similarly, if a, b, c, d be the natural order, in the order b, d, a, c, there will be three inversions: b before a, d before a, d before c.

402. In any series of integers (or letters) let two adjacent integers (or letters) be interchanged; then, the number of inversions is either increased or diminished by one.

For example, in the series 6 2 [5 1] 4 3 7, interchange 5 and 1.

We now have 6 2 [1 5] 4 3 7.

The inversions of 5 and 1 with the integers before the group are the same in both series.

The inversions of 5 and 1 with the integers after the group are the same in both series.

In the first series 5 1 is an inversion; in the second series 1 5 is not.

Hence, the interchanging of 5 and 1 diminishes the number of inversions by one.

Similarly, for any case.

403. Signs of the Terms. The principal diagonal term always has a $+$ sign.

To find the sign of any other term: Add together the number of inversions among the letters, and the number of inversions among the subscripts. If the total number is *even*, the sign of the term is $+$; if *odd*, $-$.

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$ consider the term $c_2a_3d_4b_1$. There are in $c\,a\,d\,b$ three inversions; in 2 3 4 1 three inversions; the total is six, an even number, and the sign of the term is $+$.

404. In practice the sign of a term is easily found by one of the following special rules:

RULE I. *Write the elements of the term in the natural order of letters; if the number of inversions among the subscripts is even, the sign of the term is $+$; if odd, $-$.*

RULE II. *Write the elements in the natural order of subscripts; if the number of inversions among the letters is even, the sign of the term is $+$; if odd, $-$.*

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$ consider the term $c_2a_3d_4b_1$. Writing the elements in the order of letters, we have $a_3b_1c_2d_4$. There are two inversions, viz.: 3 before 1, and 3 before 2; and the sign of the term is $+$. Or, write the elements in the order of subscripts, $b_1c_2a_3d_4$. There are two inversions, viz.: b before a, and c before a; and the sign of the term is $+$.

That these special rules give the same sign as the general rule of § 403 may be seen as follows:

Consider the term $c_2a_3d_4b_1$. Its sign is determined by the total number of inversions in the two series $\begin{matrix} c\,a\,d\,b \\ 2\,3\,4\,1 \end{matrix}$. Bring a_3 to the first position; this interchanges in the two series c and a, 2 and 3. In each series the number of inversions is increased or diminished by one (§ 402), and the total is therefore increased or diminished by an even number.

Interchange b_1 and d_4, then interchange b_1 and c_2; this brings b_1 to the second place, and the letters into the natural order. As before, the total number of inversions is changed by an even number.

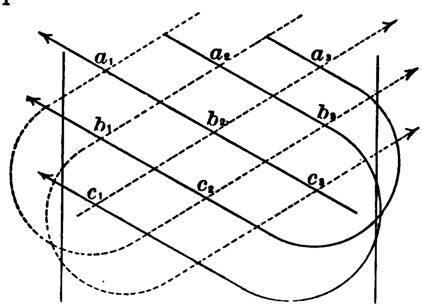
The term is now written $a_3b_1c_2d_4$, and the number of inversions differs by an even number from that found by the general rule of § 403. Hence, the sign given by Rule I. agrees with the sign given by the general rule.

405. If all the elements in any row (or column) are zero, the determinant is zero. For every term contains one of the zeros from this row (or column) (§ 400), and therefore every term of the determinant is zero.

A determinant is unchanged if the rows are changed to columns and the columns to rows. For the rules (§§ 400, 403) are unchanged if "row" is changed to "column," and "column" to "row."

Thus, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

406. A determinant of the *third order* may be conveniently expanded as follows:



Three elements connected by a full line form a positive term; three elements connected by a dotted line form a negative term. The expansion obtained from the diagram is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1,$$

which agrees with § 398.

There is no simple rule for expanding determinants of orders higher than the third.

Exercise 64.

Prove the following relations by expanding:

1. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv - \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} \equiv \begin{vmatrix} b_2 & b_1 \\ a_2 & a_1 \end{vmatrix}.$

2. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix} \equiv - \begin{vmatrix} b_1 & c_1 & a_1 \\ b_3 & c_3 & a_3 \\ b_2 & c_2 & a_2 \end{vmatrix}.$

Find the values of:

3. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 4 & 5 \end{vmatrix}.$ 4. $\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix}.$ 5. $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix}.$

6. Count the inversions in the series:

5 4 1 3 2. 7 5 1 4 3 6 2. *d a c e b*.
4 1 5 2 3. 6 5 4 2 1 3 7. *c e b d a*.

7. In the determinant $|a_1 b_2 c_3 d_4 e_5|$ find the signs of the following terms:

$a_1b_4c_5d_3e_2.$ $a_5b_1c_3d_4e_2.$ $e_1c_4a_2b_5d_3.$
$a_2b_5c_3d_1e_4.$ $b_4c_5a_1e_3d_2.$ $c_1a_5b_3e_4d_2.$

8. Write, with their proper signs, all the terms of the determinant $|a_1 b_2 c_3 d_4|$.

9. Write, with their proper signs, all the terms of the determinant $|a_1 b_2 c_3 d_4 e_5|$ which contain both a_1 and b_4; all the terms which contain both b_3 and e_5.

Expand the determinants:

10. $\begin{vmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & a & a & b \\ 0 & b & b & a \end{vmatrix}.$ 11. $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ a & a & b & b \\ b & b & a & a \end{vmatrix}.$ 12. $\begin{vmatrix} a & b & c & 0 \\ c & a & b & 0 \\ b & c & a & 0 \\ a & b & c & 1 \end{vmatrix}.$

407. Number of Terms. Consider a determinant of the nth order.

In forming a term we can take from the first row any one of n elements; from the second row any one of $n-1$ elements; and so on. From the last row we can take only the one remaining element.

Hence, the full number of terms is $n(n-1)\ldots\ldots 1$, or $\lfloor n$.

408. Interchange of Columns (or Rows). *If two adjacent columns of a determinant Δ are interchanged, the determinant thus obtained is $-\Delta$.*

For example, consider the determinants

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}, \quad \Delta' \equiv \begin{vmatrix} a_1 & a_3 & a_2 & a_4 \\ b_1 & b_3 & b_2 & b_4 \\ c_1 & c_3 & c_2 & c_4 \\ d_1 & d_3 & d_2 & d_4 \end{vmatrix}.$$

The individual elements in any row or column of Δ' are the same as those of some row or column of Δ, the only difference being in the arrangement of elements. Since every term of each determinant contains one, and only one, element from each row and column, every term of Δ' must, disregarding the sign, be a term of Δ.

Now the sign of any particular term of Δ' is found from a series (§ 404, Rule I.) in which 3 2 is the natural order. The sign of the term of Δ which contains the same elements is found from a series in which 3 2 is regarded as an inversion. Consequently every term which in Δ' has a $+$ sign has in Δ a $-$ sign, and *vice versa* (§ 402).

Therefore $\qquad \Delta' \equiv -\Delta.$

Similarly if any two adjacent columns or rows of *any* determinant are interchanged.

409. *In any determinant* Δ, *if a particular column is carried over* m *columns, the determinant obtained is* $(-1)^m\Delta$.

For, successively interchange the column in question with the adjacent column until it occupies the desired position. There will be m interchanges made, and since there will be m changes of sign (§ 408), the new determinant will be $(-1)^m\Delta$.

Similarly for a particular row.

410. *In any determinant* Δ *if any two columns are interchanged, the determinant thus obtained is* $-\Delta$.

Let there be m columns between the columns in question.

Bring the second column before the first. The second column will be carried over $m+1$ columns, and the determinant obtained is $(-1)^{m+1}\Delta$ (§ 409).

Bring the first column to the original position of the second. The first column will be carried over m columns, and the determinant obtained is $(-1)^m(-1)^{m+1}\Delta$, or $(-1)^{2m+1}\Delta$.

Since $2m+1$ is always an odd number, this is $-\Delta$.

Similarly for two rows.

Thus, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv - \begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix} \equiv \begin{vmatrix} a_3 & a_2 & a_1 \\ c_3 & c_2 & c_1 \\ b_3 & b_2 & b_1 \end{vmatrix}.$$

411. Useful Properties. *If two columns of a determinant are identical, the determinant vanishes.*

For, let Δ represent the determinant.

Interchanging the two identical columns ought to change Δ into $-\Delta$. But since the two columns are identical, the determinant is unchanged.

$$\therefore\ \Delta \equiv -\Delta, \quad 2\Delta \equiv 0, \quad \Delta \equiv 0.$$

Similarly, if two rows are identical.

412. *If all the elements in any column be multiplied by any number* m, *the determinant will be multiplied by* m.

For, every term contains one, and only one, element from the column in question. Hence every term, and consequently the whole determinant, is multiplied by m.

Similarly for a row.

Thus, $$\begin{vmatrix} ma_1 & a_2 & a_3 \\ mb_1 & b_2 & b_3 \\ mc_1 & c_2 & c_3 \end{vmatrix} \equiv m \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Again, $$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} \equiv \frac{1}{abc} \begin{vmatrix} abc & a^2 & a^3 \\ bca & b^2 & b^3 \\ cab & c^2 & c^3 \end{vmatrix} \equiv \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

413. If each of the elements in a column is the sum of two numbers, the determinant may be expressed as the sum of two determinants.

Thus, $$\begin{vmatrix} a_1+\alpha & a_2 & a_3 \\ b_1+\beta & b_2 & b_3 \\ c_1+\gamma & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & a_2 & a_3 \\ \beta & b_2 & b_3 \\ \gamma & c_2 & c_3 \end{vmatrix}.$$

For, consider any term, as $(a_1+\alpha)b_2c_3$. This may be written $a_1b_2c_3+\alpha b_2c_3$. Hence, every term of the first determinant is the sum of a term of the second determinant and a term of the third determinant. Consequently the first determinant is the sum of the other two determinants.

Similarly for any other case.

414. If the elements in any column (or row) are multiplied by any number m, and added to, or subtracted from, the corresponding elements in any other column (or row), the determinant is unchanged.

Thus, $$\begin{vmatrix} a_1 \pm ma_2 & a_2 & a_3 \\ b_1 \pm mb_2 & b_2 & b_3 \\ c_1 \pm mc_2 & c_2 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \pm \begin{vmatrix} ma_2 & a_2 & a_3 \\ mb_2 & b_2 & b_3 \\ mc_2 & c_2 & c_3 \end{vmatrix}.$$

The last determinant may be written

$$\pm m \begin{vmatrix} a_2 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_2 & c_2 & c_3 \end{vmatrix}, \text{ and therefore vanishes (§ 411).}$$

Hence, we have only the first determinant on the right-hand side.

Similarly for any other case.

This process may be applied simultaneously to two or more columns (or rows); but in this case care must be taken not to make two columns (or rows) identical (§ 411).

This last property is of great use in reducing determinants to simpler forms.

415. Examples.

$$(1)\quad \begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} \equiv \begin{vmatrix} b+c+a & a & 1 \\ c+a+b & b & 1 \\ a+b+c & c & 1 \end{vmatrix}$$

$$\equiv (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \equiv 0.$$

Begin by adding the second column to the first.

$$(2)\quad \begin{vmatrix} 14 & 15 & 11 \\ 21 & 22 & 16 \\ 23 & 29 & 17 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 11 \\ 5 & 6 & 16 \\ 6 & 12 & 17 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 & 11 \\ 5 & 3 & 16 \\ 6 & 6 & 17 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & 2 & 2 \\ 5 & 3 & 1 \\ 6 & 6 & -1 \end{vmatrix} = 2(19) = 38.$$

Begin by subtracting the third column from the first and second columns. Then take out the factor 2 (§ 312), subtract 3 times the first column from the third, and multiply out the result by § 406.

416. Factoring of Determinants. *If a determinant vanishes when for any element* **a** *we put another element* **b**, *then* **a — b** *is a factor of the determinant.*

For the expansion contains only positive integral powers of the several elements, and we can write

$$\Delta \equiv A_0 + A_1 a + A_2 a^2 + A_3 a^3 + \cdots\cdots, \tag{1}$$

where A_0, A_1, A_2, A_3, ..., are expressions which do not involve a, and will, consequently, remain unchanged when we put b for a.

Putting b for a, since Δ becomes 0 by hypothesis, we obtain

$$0 \equiv A_0 + A_1 b + A_2 b^2 + A_3 b^3 + \cdots\cdots \tag{2}$$

Subtracting (2) from (1), we obtain

$$\Delta \equiv A_1(a-b) + A_2(a^2-b^2) + A_3(a^3-b^3) \cdots\cdots$$

Since every one of the expressions $a-b$, a^2-b^2, a^3-b^3, ..., contains $a-b$ as a factor, $a-b$ is a factor of Δ.

The theorem also holds true when a and b are not elements, provided a and b enter into the expansion in positive integral powers only.

By the principle just proved, and the principle of § 411, we can resolve many determinants into factors without expanding them.

(1) Resolve into factors $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$.

The determinant vanishes when $a=b$, when $a=c$, and when $b=c$. Hence, $a-b$, $b-c$, and $c-a$ are factors. Δ is of the third degree in a, b, c, and these are easily seen to be all the factors. It remains to determine the sign before the product.

In Δ as given a^2b is $+$; in the product $(a-b)(b-c)(c-a)$ the term a^2b is $-$. Hence,

$$\Delta \equiv -(a-b)(b-c)(c-a).$$

(2) Resolve into factors $\begin{vmatrix} a^2 & a & b+c \\ b^2 & b & c+a \\ c^2 & c & a+b \end{vmatrix}$.

As in the last example $a-b$, $b-c$, $c-a$ are found to be factors There is one other factor of the first degree.

To the third column add the second; the result may be written

$$(a+b+c)\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix};$$

or, by Ex. 1, $\qquad -(a+b+c)(a-b)(b-c)(c-a).$

Exercise 65.

Show that:

1. $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} \equiv 2abc.$ 2. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \equiv 4abc.$

3. $\begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix} \equiv \begin{vmatrix} bcd & a & a^2 & a^3 \\ cda & b & b^2 & b^3 \\ dab & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix}.$

4. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}.$

Find the value of:

5. $\begin{vmatrix} 20 & 15 & 25 \\ 17 & 12 & 22 \\ 19 & 20 & 16 \end{vmatrix}.$ 6. $\begin{vmatrix} 3 & 23 & 13 \\ 7 & 53 & 30 \\ 9 & 70 & 39 \end{vmatrix}.$ 7. $\begin{vmatrix} 22 & 29 & 27 \\ 25 & 23 & 30 \\ 28 & 26 & 24 \end{vmatrix}.$

Resolve into simplest factors:

8. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$ 9. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}.$ 10. $\begin{vmatrix} a^3 & bc & 1 \\ b^3 & ca & 1 \\ c^3 & ab & 1 \end{vmatrix}.$

11. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. **12.** $\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$. **13.** $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$.

14. If all the elements on one side of a diagonal term are zeros, show that the expansion reduces to this term.

Show that:

15. $$\begin{vmatrix} a^2 - bc & a & 1 \\ b^2 - ca & b & 1 \\ c^2 - ab & c & 1 \end{vmatrix} \equiv 0.$$

16. $$\begin{vmatrix} a+2b & a+4b & a+6b \\ a+3b & a+5b & a+7b \\ a+4b & a+6b & a+8b \end{vmatrix} \equiv 0.$$

17. $$\begin{vmatrix} b^2+c^2 & ba & ca \\ ab & c^2+a^2 & cb \\ ac & bc & a^2+b^2 \end{vmatrix} \equiv 4a^2b^2c^2.$$

18. $$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} \equiv 2abc(a+b+c)^3.$$

19. $$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} \equiv x^4 + 10x^3.$$

20. $$\begin{vmatrix} a^2+1 & ba & ca & da \\ ab & b^2+1 & cb & db \\ ac & bc & c^2+1 & dc \\ ad & bd & cd & d^2+1 \end{vmatrix} \equiv a^2+b^2+c^2+d^2+1.$$

417. Minors. If one row and one column of a determinant be erased, a new determinant of order one lower than the given determinant is obtained. This determinant is called a **first minor** of the given determinant.

Similarly, by erasing two rows and two columns we obtain a **second minor**; and so on.

Thus, in the determinant $|a_1\ b_2\ c_3|$, erasing the second row and third column, we obtain the first minor $\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$. This minor is said to correspond to the element b_3, and is generally represented by Δ_{b_3}; so that, in this case, $\Delta_{b_3} \equiv \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

In general, to every element corresponds a first minor obtained by erasing the row and column in which the given element stands.

418. Theorem. *If all the elements of the first row after the first element are zeros, the determinant reduces to* $a_1\Delta_{a_1}$.

Consider the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

Every term of Δ contains one, and only one, element from the first row; and all the terms that do not contain a_1 contain one of the zeros, and therefore vanish. The terms that contain a_1 contain no other element from the first row or column, and, consequently, contain one, and only one, element from each row and column of the determinant

$$\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}, \text{ or } \Delta_{a_1}.$$

Hence, disregarding the sign, each term of Δ consists of a_1 multiplied into a term of Δ_{a_1}.

Take any particular term of Δ, as $a_1 b_4 c_3 d_2$; the sign is fixed (§ 404, Rule I.) by the number of inversions in the series 1 4 3 2; the sign of the term $b_4 c_3 d_2$ of Δ_{a_1} is fixed by the number of inversions in the series 4 3 2. Adding a_1 makes no new inversions among either the letters or the subscripts. Consequently the sign of the term in Δ is the same as the sign of the term in $a_1\Delta_{a_1}$.

Since this is true of every term of Δ, we have

$$\Delta \equiv a_1\Delta_{a_1}.$$

Similarly for any determinant of like form.

419. Terms containing an Element. From § 418 it appears that the sum of the terms which contain a_1 may be written $a_1\Delta_{a_1}$. For, no one of the terms which contain a_1 can contain any one of the elements a_2, a_3, a_4, ..., and these terms are therefore unchanged if for a_2, a_3, a_4, ... in the given determinant we put zeros.

If we carry the second column over the first, the determinant is changed to $-\Delta$. By § 418 the sum of the terms of $-\Delta$ which contain a_2 is $a_2\Delta_{a_2}$, and the sum of the corresponding terms of Δ is therefore $-a_2\Delta_{a_2}$.

In general, for the element of the pth row and qth column, we shall have to carry the pth row over $p-1$ rows, and the qth column over $q-1$ columns in order to bring the element in question to the first row and first column. The new determinant is Δ if $p+q-2$ is *even*, and is $-\Delta$ if $p+q-2$ is *odd* (§ 409). Consequently, the sum of the terms of Δ which contain the element of the pth row and qth column is the product of that element by its minor; the sign being $+$ if $p+q$ is *even*, and $-$ if $p+q$ is *odd*.

Thus, in $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$ the sum of the terms which contain c_3 is $c_3\Delta_{c_3}$.

Here $p=3$, $q=3$, and $p+q$ is *even*.

420. Co-factors. Since every term contains one element from each row and column, if we add together the sum of the terms containing a_1, the sum of the terms containing a_2, and so on, we shall obtain the whole expansion of the given determinant.

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$,

$$\Delta \equiv a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4}.$$

The expressions Δ_{a_1}, $-\Delta_{a_2}$, Δ_{a_3}, $-\Delta_{a_4}$ are called the **co-factors** of the several elements a_1, a_2, a_3, a_4, and are generally represented by A_1, A_2, A_3, A_4.

Hence, in the case of $|a_1\, b_2\, c_3\, d_4|$, we may write

$$\begin{aligned}\Delta &\equiv a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4,\\ &\equiv b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4,\\ &\equiv a_1A_1 + b_1B_1 + c_1C_1 + d_1D_1,\end{aligned}$$

and so on. Similarly for any determinant.

421. Theorem. *If the elements in any row are multiplied by the co-factors of the corresponding elements in another row, the sum of the products vanishes.*

Thus, in the determinant $|a_1\, b_2\, c_3\, d_4|$,

$$b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

No one of the co-factors B_1, B_2, B_3, B_4, contains any of the elements b_1, b_2, b_3, b_4. These co-factors will, consequently, be unaffected if in the above identity we change b_1, b_2, b_3, b_4 to a_1, a_2, a_3, a_4. This gives,

$$a_1B_1 + a_2B_2 + a_3B_3 + a_4B_4 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0.$$

Similarly for any other case.

422. Evaluation of Determinants. By using § 412, § 414, and § 420 we can readily obtain the value of any numerical determinant.

Ex. Evaluate $\begin{vmatrix} 3 & 1 & 4 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 4 & 3 & 2 & 3 \end{vmatrix}$.

From the first row subtract 3 times the second, from the third twice the second, from the fourth 4 times the second. The result is

$$\begin{vmatrix} 0 & -8 & -2 & -2 \\ 1 & 3 & 2 & 1 \\ 0 & -5 & -1 & 1 \\ 0 & -9 & -6 & -1 \end{vmatrix}$$

which, by § 420, reduces to

$$-\begin{vmatrix} -8 & -2 & -2 \\ -5 & -1 & 1 \\ -9 & -6 & -1 \end{vmatrix} \text{ or } \begin{vmatrix} 8 & 2 & 2 \\ 5 & 1 & -1 \\ 9 & 6 & 1 \end{vmatrix} = 70 \text{ (§ 406).}$$

423. Simultaneous Equations. Consider the simultaneous equations

$$a_1x + b_1y + c_1z = k_1,$$
$$a_2x + b_2y + c_2z = k_2,$$
$$a_3x + b_3y + c_3z = k_3.$$

Write the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and let A_1, A_2, B_1, B_2, etc., be the co-factors in this determinant.

Multiply the first equation by A_1, the second by A_2, the third by A_3, and add. The result is

$$(a_1A_1 + a_2A_2 + a_3A_3)x = k_1A_1 + k_2A_2 + k_3A_3,$$

since (§ 421), $\qquad b_1A_1 + b_2A_2 + b_3A_3 = 0,$

and $\qquad c_1A_1 + c_2A_2 + c_3A_3 = 0.$

Hence (§ 420), we see that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \text{ or } x = \frac{|k_1\, b_2\, c_3|}{|a_1\, b_2\, c_3|}.$$

In a similar manner,

$$y = \frac{|a_1\, k_2\, b_3|}{|a_1\, b_2\, c_3|}; \qquad z = \frac{|a_1\, b_2\, k_3|}{|a_1\, b_2\, c_3|}.$$

Similarly for any set of simultaneous equations of the first degree.

424. Elimination. To eliminate x, y, and z from the four equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \\ a_4x + b_4y + c_4z + d_4 &= 0, \end{aligned}$$

we substitute in the fourth equation the values of x, y, z found from the first three; viz. (§ 423):

$$x = -\frac{|d_1\, b_2\, c_3|}{|a_1\, b_2\, c_3|}; \quad y = -\frac{|a_1\, d_2\, c_3|}{|a_1\, b_2\, c_3|}; \quad z = -\frac{|a_1\, b_2\, d_3|}{|a_1\, b_2\, c_3|}.$$

The result is

$$-a_4 \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} - b_4 \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} - c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$+ d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

or $-a_4|b_1\, c_2\, d_3| + b_4|a_1\, c_2\, d_3| - c_4|a_1\, b_2\, d_3| + d_4|a_1\, b_2\, c_3| = 0,$

which, by § 420, may be written,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Observe that this determinant is the determinant formed by the sixteen coefficients. Of course a_1, a_2, etc., may be expressions of any kind.

Similarly for any other set of simultaneous equations.

(1) Eliminate y and z from the equations

$$\begin{aligned} 2x^2 + 3y + z &= 0, \\ 3x + 1 + y + 2z &= 0, \\ 4x^2 - 3y + 4z &= 0. \end{aligned}$$

The result is $$\begin{vmatrix} 2x^2 & 3 & 1 \\ 3x+1 & 1 & 2 \\ 4x^2 & -3 & 4 \end{vmatrix} = 0,$$

which reduces to $$8x^2 - 9x - 3 = 0.$$

(2) Eliminate x from the two equations

$$\begin{aligned} 4x^2 + 3xy + 5 &= 0, \\ 2y^2 + 3x + 4 &= 0. \end{aligned}$$

Multiply the second equation by x; we now have

$$\left.\begin{aligned} 4x^2 + 3yx + 5 &= 0 \\ 3x^2 + (2y^2+4)x &= 0 \\ 3x + (2y^2+4) &= 0 \end{aligned}\right\}.$$

Represent x^2 by u; eliminating u and x, we have

$$\begin{vmatrix} 4 & 3y & 5 \\ 3 & 2y^2+4 & 0 \\ 0 & 3 & 2y^2+4 \end{vmatrix} = 0.$$

(3) Eliminate x from the two equations

$$ax^2+bx+c=0 \text{ and } a'x+b'=0.$$

We have
$$\begin{aligned} ax^2+bx+c&=0,\\ a'x^2+c'x&=0,\\ a'x+c'&=0.\end{aligned}$$

Eliminating x^2 and x, we obtain

$$\begin{vmatrix} a & b & c \\ a' & c' & 0 \\ 0 & a' & c' \end{vmatrix}=0,$$

which reduces to
$$\frac{a}{a'^2}+\frac{c}{c'^2}=\frac{b}{a'c'}.$$

This must be the condition that there exists a value of x which satisfies both equations, since it is assumed that such is the case when we apply the process of elimination.

We have obtained, therefore, the condition that the two given equations have a common root. Cf. Ex. 39, p. 136.

Exercise 66.

1. In the determinant $|a_1\ b_2\ c_3\ d_4|$ write the co-factors of a_3, b_2, b_4, c_1, c_4, d_2, d_3.

2. Express as a single determinant

$$\begin{vmatrix} e & f & g \\ f & h & k \\ g & k & l \end{vmatrix}+\begin{vmatrix} b & e & g \\ c & f & k \\ d & g & l \end{vmatrix}+\begin{vmatrix} b & g & f \\ c & k & h \\ d & l & k \end{vmatrix}+\begin{vmatrix} b & f & e \\ c & h & f \\ d & k & g \end{vmatrix}.$$

3. Write all the terms of the following determinant which contain a:

$$\begin{vmatrix} a & 0 & b & c & b \\ a & b & c & b & 0 \\ 0 & c & b & c & 0 \\ 0 & 0 & 0 & b & c \\ b & 0 & 0 & c & b \end{vmatrix}.$$

Expand:

4. $\begin{vmatrix} a & b & b & a \\ b & a & a & b \\ a & a & b & b \\ 0 & a & b & b \end{vmatrix}$. 5. $\begin{vmatrix} 0 & d & d & d \\ a & 0 & a & a \\ b & b & 0 & b \\ c & c & c & 0 \end{vmatrix}$. 6. $\begin{vmatrix} 1 & a & a & a \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix}$.

Find the value of:

7. $\begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix}$. 8. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$. 9. $\begin{vmatrix} 2 & 1 & 3 & 4 \\ 7 & 4 & 5 & 9 \\ 3 & 3 & 6 & 2 \\ 1 & 7 & 7 & 5 \end{vmatrix}$.

Solve the equations:

10. $\left.\begin{aligned} 3x-4y+2z &= 1 \\ 2x+3y-3z &= -1 \\ 5x-5y+4z &= 7 \end{aligned}\right\}$. 11. $\left.\begin{aligned} 4x-7y+z &= 16 \\ 3x+y-2z &= 10 \\ 5x-6y-3z &= 10 \end{aligned}\right\}$.

12. $$\left.\begin{aligned} 4x+7y+3z-3w &= 9 \\ 2x-y-4z+3w &= 13 \\ 3x+2y-7z-4w &= 2 \\ 5x-3y+z+5w &= 13 \end{aligned}\right\}.$$

13. $$\left.\begin{aligned} 3x+2y+4z-w &= 13 \\ 5x+y-z+2w &= 9 \\ 2x+3y-7z+3w &= 14 \\ 4x-4y+3z-5w &= 4 \end{aligned}\right\}.$$

14. Eliminate y from the equations

$$\left.\begin{aligned} x^2+2xy+3x+4y+1 &= 0 \\ 4x+3y+1 &= 0 \end{aligned}\right\}.$$

15. Eliminate m from the equations

$$\left.\begin{aligned} m^2x-2mx^2+1 &= 0 \\ m+x^2-3mx &= 0 \end{aligned}\right\}.$$

16. Eliminate x from the equations

$$\left.\begin{aligned} ax^2+bx+c&=0\\ x^3&=1\end{aligned}\right\}.$$

17. Eliminate x from the equations

$$\left.\begin{aligned} ax^2+\ bx+\ c&=0\\ a'x^2+b'x+c'&=0\end{aligned}\right\}.$$

18. Eliminate x from the equations

$$\left.\begin{aligned} ax^2+bx+c&=0\\ x^3+qx+r&=0\end{aligned}\right\}.$$

19. Are the following equations consistent?

$$\left.\begin{aligned} 4x^2+3x+2&=0\\ 2x^2+\ x+1&=0\end{aligned}\right\}.$$

20. Are the following equations consistent?

$$\left.\begin{aligned} 3x^2+4xy+4x+1&=0\\ x-3y-7&=0\\ 2x-\ y-4&=0\end{aligned}\right\}.$$

21. If ω is one of the imaginary cube roots of 1, show that:

$$\begin{vmatrix} 1 & -\omega & \omega^2 \\ -\omega & \omega^2 & 1 \\ \omega^2 & 1 & -\omega \end{vmatrix} = -4. \qquad \begin{vmatrix} 1 & \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 & \omega \\ 1 & 1 & \omega & \omega^2 \end{vmatrix} = 3\sqrt{-3}.$$

22. Show that in any determinant there are two terms which have all but two elements alike; and that these two terms have different signs.

23. Show that the sign of a determinant is changed if the order of columns is reversed; and unchanged if the order of both columns and rows is reversed.

425. Product of Two Determinants. Consider the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 \\ a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 \\ a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}.$$

By § 413 this determinant may be expressed as the sum of 27 determinants, of which the following are types:

$$\begin{vmatrix} a_1\alpha_1 & a_2\alpha_1 & a_3\alpha_1 \\ a_1\alpha_2 & a_2\alpha_2 & a_3\alpha_2 \\ a_1\alpha_3 & a_2\alpha_3 & a_3\alpha_3 \end{vmatrix} \begin{vmatrix} a_1\alpha_1 & a_2\alpha_1 & b_3\beta_1 \\ a_1\alpha_2 & a_2\alpha_2 & b_3\beta_2 \\ a_1\alpha_3 & a_2\alpha_3 & b_3\beta_3 \end{vmatrix} \begin{vmatrix} a_1\alpha_1 & b_2\beta_1 & c_3\gamma_1 \\ a_1\alpha_2 & b_2\beta_2 & c_3\gamma_2 \\ a_1\alpha_3 & b_2\beta_3 & c_3\gamma_3 \end{vmatrix}.$$

There will be 3 determinants of the first type, 18 of the second type, and 6 of the third type. Those of the first and second types are easily seen to all vanish (§§ 411, 412). There remain the six determinants of the third type.

Consider any one of these six determinants as

$$\begin{vmatrix} c_1\gamma_1 & a_2\alpha_1 & b_3\beta_1 \\ c_1\gamma_2 & a_2\alpha_2 & b_3\beta_2 \\ c_1\gamma_3 & a_2\alpha_3 & b_3\beta_3 \end{vmatrix}.$$

This may be written

$$c_1a_2b_3\begin{vmatrix} \gamma_1 & \alpha_1 & \beta_1 \\ \gamma_2 & \alpha_2 & \beta_2 \\ \gamma_3 & \alpha_3 & \beta_3 \end{vmatrix}, \quad \text{or} \quad -c_1a_2b_3\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

It is evident that the number of interchanges required to bring the columns into the order $\alpha\,\beta\,\gamma$ is the same as the number of inversions among the letters α, β, γ; and also the same as the number of inversions among the letters a, b, c. Hence the sign will be $+$ if that number is even, and $-$ if the number is odd. The sign before $c_1a_2b_3$ is therefore the sign of this term in the determinant $|a_1\, b_2\, c_3|$ (§ 404, Rule II.).

Since the preceding is true for each one of the six determinants of the third type, the given determinant is the product of the expansion of $|a_1 b_2 c_3|$ by the determinant $|\alpha_1 \beta_2 \gamma_3|$, and is one of the forms in which the product

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

may be written.

The above proof is perfectly general, and may be extended to the product of any two determinants.

(1) Write as a determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

The result is $\begin{vmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{vmatrix}$.

(2) Write as a determinant the product

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \times \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

The result is $\begin{vmatrix} X & Y & Z \\ Z & X & Y \\ Y & Z & X \end{vmatrix}$, where

$$X = ax + by + cz, \quad Y = cx + ay + bz, \quad Z = bx + cy + az.$$

426. The notation

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{Vmatrix} = 0$$

is used to denote that the four determinants obtained by omitting one of the four columns all vanish.

Exercise 67.

1. Show that $\begin{vmatrix} a & b & 0 \\ c & 0 & c \\ 0 & b & a \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ c & 0 & c \\ b & a & 0 \end{vmatrix} \equiv -4a^2b^2c^2.$

2. Express as a single determinant

$$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

3. Express as a single determinant

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \times \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix},$$

and thence resolve the first determinant into its simplest factors.

4. Express as a single determinant

$$\begin{vmatrix} a+bi & -c+di \\ c+di & a-bi \end{vmatrix} \times \begin{vmatrix} \alpha+\beta i & -\gamma+\delta i \\ \gamma+\delta i & \alpha-\beta i \end{vmatrix},$$

where $i=\sqrt{-1}$; and thence prove Euler's theorem, viz.: *the product of two sums of four squares can itself be expressed as the sum of four squares.*

5. Show that $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2.$

NOTE. The student who wishes to pursue the study of determinants further is referred to the treatises of *Muir*, published by Macmillan, and *Hanus*, published by Ginn & Co.

CHAPTER XXVI

GENERAL PROPERTIES OF EQUATIONS

We now resume the subject of equations where we left it at the end of Chapter XIII. Before proceeding further, however, the student should carefully review §§ [illegible] to 83.

427. Definitions. A function of a variable x has already been defined (§ 357) to be any expression which changes value when x changes value. Any expression which involves x is, in general, a function of x. If x is involved only in powers and roots, the expression is an **algebraic function** of x.

Thus x^2, $\sqrt{x^2-x}$, $\frac{1}{x^2-4}$, e^x, $\log x$, are functions of x, the first three being algebraic functions of x.

428. An algebraic function of x is **rational** as regards x if x is involved only in powers; that is, not in roots. It is **rational** and **integral** as regards x, if x is involved only in positive integral powers; that is, in numerators and not in denominators.

Thus,
$$\frac{1}{x^2},\quad x^{-3},\quad \frac{1}{4x-3},\quad \frac{x}{x^2-a^2},\quad \frac{3x^2+4}{5x^3-3x+2}$$
are rational but not integral functions of x, while
$$4x^2-3x-7,\quad ax^3-bx^2+cx+d,$$
are rational integral functions of x.

429. Quantics. A function which is rational and integral with regard to all the variables involved is called a **quantic**. We shall consider in this chapter only functions of one variable, and by quantic will be meant a rational integral

function of one variable, unless it is expressly stated that several variables are involved.

NOTE. The term quantic is generally applied only to homogeneous expressions like $ax^2 + bxy + cy^2$. This expression is obtained from $ax^2 + bx + c$ by putting $\frac{x}{y}$ for x, and multiplying through by y^2. The theory of the two expressions is precisely the same, and we shall therefore extend the term *quantic* to include expressions like $ax^2 + bx + c$, $ax^3 + bx^2 + cx + d$, etc.

The **degree** of a quantic involving only one variable x is the same as the exponent of the highest power of x involved in the quantic (§ 82).

A quantic of the first degree is called a *linear function;* quantics of higher degrees are called *quadratics*, *cubics*, *quartics*, *quintics*, etc.

430. General Form. Any quantic of the nth degree in which x is the variable may be written in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots a_{n-1}x + a_n,$$

where $a_0, a_1, \ldots\ldots, a_n$ are coefficients which do not involve x. Some of these coefficients may be zero, and in that case the corresponding terms will be wanting.

The *coefficients* may be real, imaginary, surd, or rational expressions. We shall, in general, consider only quantics which have real and rational coefficients. The student will readily see what properties of such quantics are possessed by quantics with surd or imaginary coefficients.

431. Abbreviations. For brevity a quantic involving x is often represented by $f(x)$, $F(x)$, $\phi(x)$, or some similar notation.

The value of the quantic $f(x)$, when we put a for x, is represented by $f(a)$.

Thus, if $\quad f(x) \equiv 2x^3 - x^2 + 3x + 4,$

we have $\quad f(2) = 2(2)^3 - 2^2 + 3(2) + 4 = 16 - 4 + 6 + 4 = 22.$

4

432. Equations. Every rational integral equation (§ 328) involving but one variable x can, by transposing all the terms to the first member, be made to assume the form $f(x) = 0$, where $f(x)$ is a quantic involving the one variable x. The theory of this quantic and that of the corresponding equation are closely related, and we shall develop the two together.

The **roots** of the equation $f(x) = 0$ are those values of x which cause the quantic $f(x)$ to vanish. These roots are also called the roots of the quantic.

The **degree** of the equation $f(x) = 0$ is the same as that of the quantic $f(x)$.

433. Divisibility of Quantics. Theorem I. *If* $\mathbf{h}$ *is a root of the equation* $\mathbf{f}(\mathbf{x}) = 0$, *the quantic* $\mathbf{f}(\mathbf{x})$ *is divisible by* $\mathbf{x} - \mathbf{h}$.

For example, consider the quantic

$$f(x) \equiv ax^3 + bx^2 + cx + d.$$

Now, since h is a root of the equation $f(x) = 0$, we have

$$0 = ah^3 + bh^2 + ch + d.$$

Subtracting,

$$f(x) \equiv a(x^3 - h^3) + b(x^2 - h^2) + c(x - h).$$

Each of the expressions $x - h$, $x^2 - h^2$, $x^3 - h^3$, is divisible by $x - h$, and consequently $f(x)$ is divisible by $x - h$. Similarly for any other quantic. Cf. § 416.

434. Theorem II. *Conversely, if a quantic* $\mathbf{f}(\mathbf{x})$ *is divisible by* $\mathbf{x} - \mathbf{h}$, *then* $\mathbf{h}$ *is a root of the equation* $\mathbf{f}(\mathbf{x}) = 0$.

For, if $\phi(x)$ be the quotient obtained by dividing $f(x)$ by $x - h$, we have

$$f(x) \equiv (x - h)\,\phi(x),$$

and the equation $f(x) = 0$ may be written

$$(x - h)\,\phi(x) = 0,$$

of which h is evidently a root (§ 84).

435. Synthetic Division. Let the quantic

$$3x^5 - 4x^4 + x^3 - 12x^2 + 3x + 6$$

be divided by $x - 2$.

The work is as follows:

$$
\begin{array}{l}
3x^5 - 4x^4 + x^3 - 12x^2 + 3x + 6 \;\underline{|\, x - 2 \qquad\qquad\qquad\qquad\qquad} \\
\underline{3x^5 - 6x^4} \qquad\qquad\qquad\qquad\qquad\quad 3x^4 + 2x^3 + 5x^2 - 2x - 1 \\
\quad + 2x^4 + x^3 \\
\quad \underline{+ 2x^4 - 4x^3} \\
\qquad\quad + 5x^3 - 12x^2 \\
\qquad\quad \underline{+ 5x^3 - 10x^2} \\
\qquad\qquad\quad - 2x^2 + 3x \\
\qquad\qquad\quad \underline{- 2x^2 + 4x} \\
\qquad\qquad\qquad\quad - x + 6 \\
\qquad\qquad\qquad\quad \underline{- x + 2} \\
\qquad\qquad\qquad\qquad\quad 4
\end{array}
$$

The work may be abridged by omitting the powers of x, and writing only the coefficients.

We now have

$$
\begin{array}{l}
3 - 4 + 1 - 12 + 3 + 6 \;\underline{|\, 1 - 2 \qquad\qquad\qquad} \\
\underline{3 - 6} \qquad\qquad\qquad\quad 3 + 2 + 5 - 2 - 1 \\
\quad + 2 + 1 \\
\quad \underline{+ 2 - 4} \\
\qquad + 5 - 12 \\
\qquad \underline{+ 5 - 10} \\
\qquad\quad - 2 + 3 \\
\qquad\quad \underline{- 2 + 4} \\
\qquad\qquad - 1 + 6 \\
\qquad\qquad \underline{- 1 + 2} \\
\qquad\qquad\qquad 4
\end{array}
$$

But the operation may be still further abridged. As the first term of the divisor is unity, the first term of each remainder is the next term of the quotient, and we need not write the quotient. Second, we need not bring down the several terms of the dividend. Third, we need not write the first terms of the partial products.

The work is now as follows:

$$\begin{array}{rrrrrrl}
3 & -4 & +1 & -12 & +3 & +6 & \underline{|\,1-2} \\
\underline{-6} & & & & & & \\
+2 & & & & & & \\
& \underline{-4} & & & & & \\
& +5 & & & & & \\
& & \underline{-10} & & & & \\
& & -\;\;2 & & & & \\
& & & \underline{+4} & & & \\
& & & -1 & & & \\
& & & & \underline{+2} & & \\
& & & & +4 & &
\end{array}$$

Omitting the first term of the divisor, which is now useless, changing -2 to $+2$, and adding, instead of subtracting, we have, raising the terms and bringing down the first coefficient,

$$\begin{array}{l}
3-4+1-12+3+6\,\lfloor\underline{2} \\
\underline{\;\;\;+6+4+10-4-2} \\
3+2+5-\;\;2-1+4
\end{array}$$

The last term below the line gives us the remainder, the preceding terms the coefficients of the quotient. In this particular problem the quotient is $3x^4+2x^3+5x^2-2x-1$, and the remainder is 4.

This method is called the method of **Synthetic Division.** For the application of this method to the division of any quantic by $x-h$ we have the following rule:

Write the coefficients a, b, c, *etc., in a horizontal line.*

Bring down the first coefficient a.

Multiply a *by* h, *and add the product to* b.

Multiply the sum so obtained by h, *and add the product to* c.

Continuing this process, the last sum will be the remainder, and the preceding sums the coefficients of the quotient.

REMARK. If there are any powers of x missing, their places are to be supplied by zero coefficients.

Ex. Divide $2x^4 - 6x^2 + 5x - 2$ by $x - 3$.

$$\begin{array}{l|l} 2+0-\ 6+\ 5-\ \ 2 & 3 \\ \quad +6+18+36+123 & \\ \hline 2+6+12+41+121 & \end{array}$$

The quotient is $2x^3 + 6x^2 + 12x + 41$, and the remainder 121.

436. Value of a Quantic. By the principles of division it is evident that the operation of dividing a given quantic $f(x)$ by $x - h$ can be carried on until the remainder does not involve x. Represent the quotient by $\phi(x)$, and the remainder by R. Then, we have

$$f(x) \equiv (x - h)\phi(x) + R.$$

Putting h for x,

$$f(h) = 0 + R.$$

Hence, the value which a quantic f(x) *assumes when we put* h *for* x *is equal to the last remainder obtained in the operation of dividing* f(x) *by* x − h.

This remainder, and, consequently, the value of the quantic, may be easily calculated by the method of synthetic division.

The truth of the above theorem may also be shown by another method, which has the advantage of showing the form of the quotient and remainder.

Take, for example, the quantic

$$ax^4 + bx^3 + cx^2 + dx + e.$$

Divide the quantic by $x - h$. The work is as follows:

$$\begin{array}{ccccc|c} a & b & c & d & e & h \\ & ah & Bh & Ch & Dh & \\ \hline a & B & C & D & R & \end{array}$$

where
$$\begin{aligned} B &= ah + b, \\ C &= Bh + c = ah^2 + bh + c, \\ D &= Ch + d = ah^3 + bh^2 + ch + d, \\ R &= Dh + e = ah^4 + bh^3 + ch^2 + dh + e. \end{aligned}$$

The remainder R is evidently the value which the quantic assumes when we put h for x.

The quotient is

$$ax^3 + (ah + b)x^2 + (ah^2 + bh + c)x + (ah^3 + bh^2 + ch + d).$$

Similarly for any quantic.

Exercise 68.

Find the quotient and remainder obtained by dividing each of the following quantics by the divisor opposite it.

1. $x^4 - 3x^3 - x^2 + 2x - 1$ $\quad x - 2.$
2. $x^4 - 3x^2 + 2x - 7$ $\quad x - 3.$
3. $2x^4 + 3x^3 - 8x^2 - 7x - 10$ $\quad x - 2.$
4. $3x^4 + 2x^2 - 6x + 50$ $\quad x + 3.$
5. $ax^3 + 3bx^2 + 3cx + d$ $\quad x + h.$

Are the following numbers roots of the equations opposite them (§ 434)?

6. (3) $\quad x^4 + x^2 - 6x + 2 = 0.$
7. (− 7) $\quad x^4 + 7x^3 + 21x + 147 = 0.$
8. (0.3) $\quad x^4 - 2.3x^3 + 3.6x^2 + 4.9x + 1.2 = 0.$

Find the value of the following expressions when for x we put the number in parentheses:

9. $3x^3 + 2x^2 - 6x + 1$ $\quad (-3).$
10. $2x^4 + 6x^2 - 9x - 5$ $\quad (6).$
11. $x^5 + 7x^3 - 2x^2 - 49$ $\quad (-4).$
12. $x^4 + 6x^3 - 7x^2 - 3x + 1$ $\quad (-0.2).$

437. Number of Roots. We shall assume that every rational integral equation has at least one root. The proof of this truth is beyond the scope of the present chapter.*

Let $f(x)=0$ be a rational integral equation of the nth degree. This equation has, by assumption, at least one root. Let α_1 be a root.

Then, by § 433, $f(x)\equiv(x-\alpha_1)f_1(x)$,

where $f_1(x)$ is a quantic of degree $n-1$.

The equation $f_1(x)=0$ must, by assumption, have a root. Let α_2 be a root.

Then, by § 433, $f_1(x)\equiv(x-\alpha_2)f_2(x)$,

where $f_1(x)$ is a quantic of degree $n-2$.

Continuing this process, we see that at each step the degree of the quotient is diminished by one. Hence, we can find n factors $x-\alpha_1,\ x-\alpha_2 \ldots\ldots x-\alpha_n$. The last quotient will not involve x, and is readily seen to be a_0, the coefficient of x^n in $f(x)$.

Now,
$$\begin{aligned} f(x) &\equiv (x-\alpha_1)f_1(x) \\ &\equiv (x-\alpha_1)(x-\alpha_2)f_2x \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\equiv a_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n), \end{aligned}$$

so that the equation $f(x)=0$ may be written

$$a_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)=0,$$

which evidently holds true if x has any one of the n values $\alpha_1,\ \alpha_2 \ldots\ldots \alpha_n$.

It follows, then, that if *every* rational integral equation has one root, *an equation of the nth degree has n roots.*

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* See Burnside and Panton, *Theory of Equations*, 2d ed., Art. 195; Briot et Bouquet, *Fonctions Elliptique*, Art. 23.

438. Linear Factors. The factors $x-a_1$, $x-a_2$ ..... $x-a_n$ are linear functions of x (§ 429).

When $f(x)$ is written in the form

$$a_0(x-a_1)(x-a_2)\ .....\ (x-a_n),$$

it is said to be resolved into its linear factors.

From § 437 it follows that a quantic can be resolved into linear factors in only one way.

To resolve a quantic $f(x)$ into linear factors is evidently equivalent to solving the equation $f(x)=0$.

439. Multiple Roots. The n roots of an equation of the nth degree are not necessarily all different.

Thus, the equation $x^3-7x^2+15x-9=0$ may be written $(x-1)(x-3)(x-3)=0$, and the roots are seen to be 1, 3, 3.

The root 3, and the corresponding factor $x-3$, occurs twice; hence, 3 is said to be a *double root.* When a root occurs three times, it is called a *triple root;* four times, a *quadruple root;* and so on.

Any root which occurs more than once is a **multiple root.**

440. Roots Given. When all the roots of an equation are given, the equation can at once be written.

Ex. Write the equation of which the roots are 1, 2, 4, −5.

The equation is $\quad (x-1)(x-2)(x-4)(x+5)=0,$

or $\quad x^4-2x^3-21x^2+62x-40=0.$

441. Solutions by Trial. When all the roots of an equation but two can be found by trial, the equation can be readily solved by the process of § 437. The work can be much abbreviated by employing the method of synthetic division (§ 435). Cf. § 140.

Ex. Solve the equation

$$x^4 - 3x^3 - 6x^2 + 14x + 12 = 0.$$

Try $+1$ and -1. Substituting these values for x, we obtain

$$1 - 3 - 6 + 14 + 12 = 0,$$
$$1 + 3 - 6 - 14 + 12 = 0,$$

which are both false, so that neither $+1$ nor -1 is a root.

Try 2. Dividing by $x - 2$,

$$\begin{array}{rrrrr|l} 1 & -3 & -6 & +14 & +12 & \underline{2} \\ & +2 & -2 & -16 & -4 & \\ \hline 1 & -1 & -8 & -2 & +8 & \end{array}$$

we see that 2 is not a root.

Try 3. Dividing by $x - 3$,

$$\begin{array}{rrrrr|l} 1 & -3 & -6 & +14 & +12 & \underline{3} \\ & +3 & +0 & -18 & -12 & \\ \hline 1 & +0 & -6 & -4 & 0 & \end{array}$$

we see that 3 is a root. The quotient is $x^3 - 6x - 4$.

In this quotient try 3 again. Dividing by $x - 3$,

$$\begin{array}{rrrr|l} 1 & +0 & -6 & -4 & \underline{3} \\ & +3 & +9 & +9 & \\ \hline 1 & +3 & +3 & +5 & \end{array}$$

we see that 3 is not again a root.

Try -2. Dividing by $x + 2$,

$$\begin{array}{rrrr|l} 1 & +0 & -6 & -4 & \underline{-2} \\ & -2 & +4 & +4 & \\ \hline 1 & -2 & -2 & 0 & \end{array}$$

we see that -2 is a root. The quotient is $x^2 - 2x - 2$.

Hence the given equation may be written

$$(x - 3)(x + 2)(x^2 - 2x - 2) = 0.$$

Therefore one of the three factors must vanish.

If $x - 3 = 0$, $x = 3$; if $x + 2 = 0$, $x = -2$; if $x^2 - 2x - 2 = 0$, solving this quadratic, we find $x = 1 + \sqrt{3}$ or $x = 1 - \sqrt{3}$. Hence the four roots of the given equation are

$$3, \quad -2, \quad 1 + \sqrt{3}, \quad 1 - \sqrt{3}.$$

Exercise 69.

Solve the equations:

1. $x^3 - 7x^2 + 16x - 12 = 0.$
2. $x^3 + 9x^2 + 2x - 48 = 0.$
3. $x^3 - 4x^2 - 8x + 8 = 0.$
4. $x^3 - 5x^2 - 2x + 24 = 0.$
5. $x^3 + 2x^2 + 4x + 3 = 0.$
6. $x^3 - 6x^2 + 6x + 99 = 0.$
7. $6x^3 - 29x^2 + 14x + 24 = 0.$
8. $2x^3 + 3x^2 - 13x - 12 = 0.$
9. $x^4 - 15x^2 - 10x + 24 = 0.$
10. $x^4 + 5x^3 - 5x^2 - 45x - 36 = 0.$
11. $x^4 + 4x^3 - 29x^2 - 156x + 180 = 0.$
12. $x^4 - 5x^3 - 2x^2 + 12x + 8 = 0.$
13. $6x^4 - 5x^3 - 30x^2 + 20x + 24 = 0.$
14. $4x^4 + 8x^3 - 23x^2 - 7x + 78 = 0.$

Form the equations which have the following roots:

15. $2, 6, -7.$
16. $2, 4, -3.$
17. $2, 0, -2.$
18. $2, 1, -2, -1.$
19. $5, 3+\sqrt{-1}, 3-\sqrt{-1}.$
20. $2, \frac{1}{2}, 2, -\frac{1}{2}.$
21. $2, 3, -2, -3, -6.$
22. $\frac{1}{3}, \frac{2}{3}, -\frac{1}{2}, -\frac{3}{2}.$
23. $3+\sqrt{2}, 3-\sqrt{2}, 2+\sqrt{3}, 2-\sqrt{3}.$
24. $0.2, 0.125, -0.4.$
25. $0.3, -0.2, -\frac{1}{20}, -\frac{6}{5}.$
26. $2+\sqrt{-1}, 2-\sqrt{-1}, 1+2\sqrt{-1}, 1-2\sqrt{-1}.$

442. Relations between the Roots and the Coefficients. The quadratic equation of which the roots are α and β is (§ 153)

$$(x-\alpha)(x-\beta)=0,$$

or, multiplying out,

$$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

The cubic equation of which the roots are α, β, γ is

$$(x-\alpha)(x-\beta)(x-\gamma)=0,$$

or $\quad x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0.$

The biquadratic equation of which the roots are α, β, γ, δ is

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0,$$

or

$$x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2$$
$$-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x+\alpha\beta\gamma\delta=0.$$

And so on.

Take any equation in which the highest power of x has the coefficient unity. From the above we have the following relations between the roots and the coefficients:

The coefficient of the *second* term, with its sign changed, is equal to the sum of the roots.

The coefficient of the *third* term is equal to the sum of all the products that can be formed by taking the roots *two* at a time.

The coefficient of the *fourth* term, with its sign changed, is equal to the sum of all the products that can be formed by taking the roots *three* at a time.

The coefficient of the *fifth* term is equal to the sum of all the products that can be formed by taking the roots *four* at a time; and so on.

If the number of roots is *even*, the last term is equal to the product of all the roots. If the number of roots is

odd, the last term, *with its sign changed*, is equal to the product of all the roots.

Observe that the sign of the coefficient is changed when an odd number of roots are taken to form a product; that the sign is unchanged when an even number of roots are taken to form a product.

443. By dividing the equation through by the coefficient of the highest power of x, any rational integral equation whatever can be reduced to a form in which the coefficient of the highest power of x is unity.

We shall write an equation reduced to this form, called the "*p form*," as follows:

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots\cdots p_{n-1}x + p_n = 0.$$

Let α, β, γ, etc., be the roots of this equation. Represent by $\Sigma\alpha$ the sum of the roots, by $\Sigma\alpha\beta$ the sum of all the products that can be formed by taking the roots two at a time; and so on.

From § 442 we now have

$$\begin{array}{ll}
\Sigma\alpha \quad = -p_1, & p_1 = -\Sigma\alpha, \\
\Sigma\alpha\beta \; = +p_2, & p_2 = +\Sigma\alpha\beta, \\
\Sigma\alpha\beta\gamma = -p_3, & p_3 = -\Sigma\alpha\beta\gamma, \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\
\alpha\beta\gamma\delta \cdots\cdots = (-1)^n p_n. & p_n = (-1)^n \alpha\beta\gamma \cdots\cdots
\end{array}$$

Ex. Let α, β, γ be the roots of the equation

$$x^3 - 7x^2 - 9x + 4 = 0.$$

Then,
$$\begin{aligned}
\Sigma\alpha &= \alpha + \beta + \gamma = 7, \\
\Sigma\alpha\beta &= \beta\gamma + \gamma\alpha + \alpha\beta = -9, \\
\alpha\beta\gamma &= -4.
\end{aligned}$$

The relations between the roots and the coefficients of an equation do not assist us to solve the equation. In every case we are brought at last to the original equation.

Thus, in the equation

$$x^3 - 7x^2 - 9x + 4 = 0,$$

we have
$$\alpha + \beta + \gamma = 7,$$
$$\beta\gamma + \gamma\alpha + \alpha\beta = -9,$$
$$\alpha\beta\gamma = -4.$$

Eliminating β and γ, we have to solve the equation

$$\alpha^3 - 7\alpha^2 - 9\alpha + 4 = 0;$$

that is, we have to solve the given equation.

444. Symmetric Functions of the Roots. The expressions $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, ..., are examples of **symmetric functions** of the roots (§ 152). Any expression which involves all the roots, the roots all entering to similar powers and with similar coefficients, is a symmetric function of the roots.

From the relations $\Sigma\alpha = -p_1$, $\Sigma\alpha\beta = +p_2$, $\Sigma\alpha\beta\gamma = -p_3$, ..., the value of any symmetric function of the roots of a given equation may be found in terms of the coefficients.

If α, β, γ are the roots of the equation

$$x^3 - 4x^2 + 6x - 5 = 0,$$

we may calculate the values of symmetric functions of the roots as follows:

We have
$$\alpha + \beta + \gamma = 4, \tag{1}$$
$$\beta\gamma + \gamma\alpha + \alpha\beta = 6, \tag{2}$$
$$\alpha\beta\gamma = 5. \tag{3}$$

(1) $\Sigma\alpha^2 \equiv \alpha^2 + \beta^2 + \gamma^2$.

Square (1), $\quad \alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta = 16$

But, by (2), $\quad 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta = 12$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = 4$$

(2) $\Sigma\alpha^2\beta \equiv \alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta$.

Multiply (1) by (2), $\quad \Sigma\alpha^2\beta + 3\alpha\beta\gamma = 24$

But, by (3), $\quad 3\alpha\beta\gamma = 15$

$$\therefore \Sigma\alpha^2\beta = 9$$

(3) $\Sigma\alpha^3 \equiv \alpha^3 + \beta^3 + \gamma^3.$

Multiply $\alpha^2 + \beta^2 + \gamma^2$ by $\alpha + \beta + \gamma$;

the result is $\alpha^3 + \beta^3 + \gamma^3 + \Sigma\alpha^2\beta = 16$

But $\Sigma\alpha^2\beta = 9$

$$\therefore\ \alpha^3 + \beta^3 + \gamma^3 = 7$$

And so on. Cf. § 152.

445. By the aid of the preceding sections we can find the condition that a given relation should exist among the roots of an equation.

Find the condition that the roots of the equation

$$x^3 + px^2 + qx + r = 0$$

shall be in geometrical progression.

Let β be the mean root. Then,

$$\alpha + \beta + \gamma = -p, \quad (1)$$

$$\beta\gamma + \gamma\alpha + \alpha\beta = q, \quad (2)$$

$$\alpha\beta\gamma = -r, \quad (3)$$

and $$\beta^2 = \gamma\alpha. \quad (4)$$

From (2) and (4),

$$\beta\gamma + \alpha\beta + \beta^2 = q,$$

or, by (1), $$-p\beta = q.$$

$$\therefore\ \beta = -\frac{q}{p}.$$

Substituting in (3), $$\left(-\frac{q}{p}\right)^3 = -r,$$

or $q^3 = p^3 r$, the required condition.

446. Imaginary Roots. If an imaginary number is a ro[ot] of an equation with real coefficients, the conjugate imag[i]nary (§ 176) is also a root.

Let $\alpha + \beta i$, where $i = \sqrt{-1}$, be a root of the equation

$$a_0x^n + a_1x^{n-1} + \cdots\cdots a_n = 0,$$

the coefficients being real.

Put $\alpha + \beta i$ for x in the left member of this equation, and expand the powers of $\alpha + \beta i$ by the binomial theorem. All the terms which do not contain i, and all the terms which contain even powers of i, will be real; all the terms which contain odd powers of i will be imaginary. Representing the real part of the result by P, and the imaginary part of the result by Qi, we have (§ 432), since $\alpha + \beta i$ is a root,

$$P + Qi = 0,$$

and therefore $P = 0$ and $Q = 0$ (§ 179).

Now put $\alpha - \beta i$ for x in the given equation. The result may be obtained from the former result by changing i to $-i$. The even powers of i will be unchanged while the odd powers will have their signs changed. The real part will therefore be unchanged, and the imaginary part changed only in sign. The result is

$$P - Qi,$$

which vanishes, since by the preceding $P = 0$ and $Q = 0$.

Therefore $\alpha - \beta i$ is a root of the given equation (§ 432).

This theorem is generally stated as follows: *Imaginary roots enter equations in pairs.*

The above proof will be more readily understood if applied to an equation of the third or fourth degree.

Corresponding to a pair of imaginary roots, we shall have the factors $\qquad x - \alpha - \beta i, \;\; x - \alpha + \beta i.$

The product of these,

$$(x - \alpha)^2 + \beta^2,$$

is positive, provided x is real. Hence, corresponding to a pair of imaginary roots, we have a factor of the second degree, which for real values of x does not change sign (§ 180).

Exercise 70.

1. Form the equation of which the roots are:

$$2,\ 4,\ -3; \qquad 3,\ -2,\ -4.$$

If α, β, γ are the roots of $x^3 - 5x^2 + 4x - 3 = 0$, f the value of:

2. $\Sigma\alpha^2$.
3. $\Sigma\alpha^2\beta$.
4. $\Sigma\alpha^3$.
5. $\Sigma\alpha^2\beta\gamma$.
6. $\Sigma\alpha^2\beta^2$.
7. $\Sigma\alpha^3\beta$.
8. $\Sigma\alpha^4$.
9. $\Sigma\alpha^3\beta\gamma$.
10. $\Sigma\alpha^2\beta^2\gamma$.

If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find terms of the coefficients the values of:

11. $\Sigma\alpha^2$.
12. $\Sigma\alpha^2\beta$.
13. $\Sigma\alpha^3$.
14. $\Sigma\alpha^2\beta^2$.
15. $\Sigma\alpha^4$.
16. $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$.
17. $\dfrac{\beta\gamma}{\alpha} + \dfrac{\gamma\alpha}{\beta} + \dfrac{\alpha\beta}{\gamma}$.
18. $\dfrac{\beta^2+\gamma^2}{\beta\gamma} + \dfrac{\gamma^2+\alpha^2}{\gamma\alpha} + \dfrac{\alpha^2+\beta^2}{\alpha\beta}$.
19. $\dfrac{\beta^2+\gamma^2}{\beta+\gamma} + \dfrac{\gamma^2+\alpha^2}{\gamma+\alpha} + \dfrac{\alpha^2+\beta^2}{\alpha+\beta}$.

In the equation $x^3 + px^2 + qx + r = 0$, find the condi tion that:

20. One root is the negative of one of the other tw roots.

21. One root is double another.

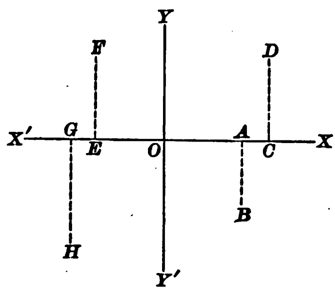
22. The three roots are in arithmetical progression.

23. The three roots are in harmonical progression.

GRAPHICAL REPRESENTATION OF FUNCTIONS.

The investigation of the changes in the value of $f(x)$ corresponding to changes in the value of x is much facilitated by using the system of graphical representation explained in the following sections.

447. Co-ordinates. Let $X'X$ and $Y'Y$ be two perpendicular straight lines drawn in a plane, intersecting at O.



The lines $X'X$ and $Y'Y$ are called **axes of reference**; the point O is called the **origin.**

Distances measured from O along $X'X$, as OA, OC, OE, and OG, are called **abscissas**; distances measured from $X'X$ parallel to $Y'Y$, as AB, CD, EF, and GH, are called **ordinates.**

Abscissas are considered positive if measured to the right; negative, if measured to the left. Ordinates are considered positive if measured upwards; negative, if measured downwards.

Thus, OA, OC, CD, and EF are positive; OE, OG, AB, and GH are negative.

An abscissa is generally represented by x, an ordinate is generally represented by y.

The abscissa and ordinate of any point are called the **co-ordinates** of that point. Thus the co-ordinates of B are OA and AB.

The co-ordinates of a point are written thus: (x, y).

Thus, (7, 4) is the point of which the abscissa is 7 and the ordinate 4.

The axis $X'X$ is called the **axis of abscissas,** or the **axis of x**; the axis $Y'Y$, the **axis of ordinates,** or the **axis of y.**

448. It is evident that if a point B is given, its co-ordinates referred to given axes may be found by drawing the ordinate and measuring the distances OA and AB.

Conversely, if the co-ordinates of a point are given, the point may be readily constructed.

Thus, to construct the point $(7, -4)$, a convenient length is taken as a unit of length. A distance of 7 units is laid off on OX to the *right* from O to A. At A a perpendicular to $X'X$ is drawn *downwards*, of length 4 units, to B. Then B is the required point.

Ex. Construct the points $(3, 2)$; $(5, 4)$; $(6, -3)$; $(-4, -3)$; $(-4, 2)$; $(-3, -5)$; $(4, -3)$.

449. Graph of a Function. Let $f(x)$ be any function of x, where x is a variable. Put $y = f(x)$; then y is a new variable connected with x by the relation $y = f(x)$. If $f(x)$ is a rational integral function of x, it is evident that to every value of x corresponds one, and only one, value of y.

If different values of x be laid off as abscissas, and the corresponding values of $f(x)$ as ordinates, the points thus obtained will all lie on a line; this line will generally be a curved line, or, as it is briefly called, a *curve*. This curve is called the **graph** of the function $f(x)$; it is also called the **locus** of the equation $y = f(x)$.

We proceed to construct the graphs of several functions.

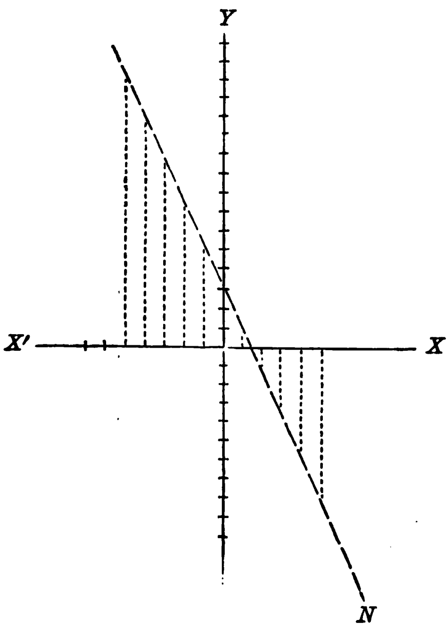
Remark. In constructing, or *plotting*, as it is called, the graph of a function, the student will find it convenient to use the paper called plotting, or co-ordinate, paper. This is ruled in small squares, and therefore saves much labor.

(1) Construct the graph of $3-2x$.

Put $y=3-2x$. The following table is readily computed:

If	$x=1$,	$y=\ \ 1$.	If	$x=-1$,	$y=\ \ 5$.
"	$x=2$,	$y=-1$.	"	$x=-2$,	$y=\ \ 7$.
"	$x=3$,	$y=-3$.	"	$x=-3$,	$y=\ \ 9$.
"	$x=4$,	$y=-5$.	"	$x=-4$,	$y=11$.
"	$x=5$,	$y=-7$.	"	$x=-5$,	$y=13$.

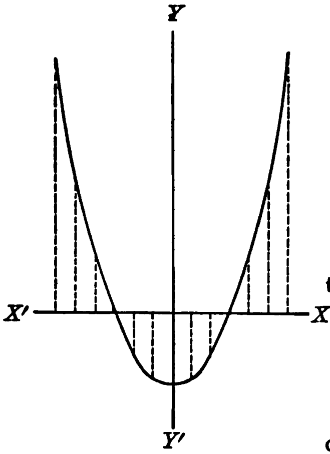
Constructing the above points, it appears that the graph of the function $3-2x$ is the straight line MN.



In general, where the equation $y=f(x)$ contains only the first powers of x and y, the locus will be a straight line.

(2) Plot the graph of $\frac{1}{2}x^2 - 4$.

Putting $y = \frac{1}{2}x^2 - 4$, we readily compute the following table:

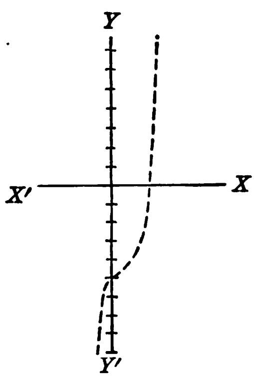


$x = 0,$	$y = -4.$
$x = \pm 1,$	$y = -3.5$
$x = \pm 2,$	$y = -2.$
$x = \pm 3,$	$y = +0.5$
$x = \pm 4,$	$y = +4.$
$x = \pm 5,$	$y = +8.5$
$x = \pm 6,$	$y = +14.$

Plotting these points, we obtain the curve here given.

(3) Plot the graph of
$$x^3 - x^2 + x - 5.$$

Putting $y = x^3 - x^2 + x - 5$, we compute the following table:



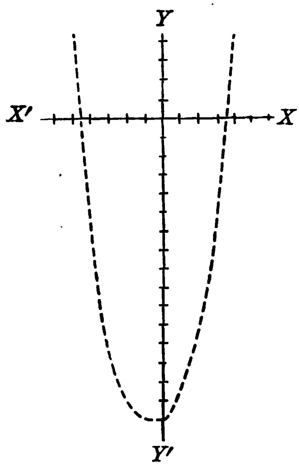
If x is	y is
0.5,	$-4.625.$
1.0,	$-4.000.$
1.5,	$-2.375.$
2.0,	$+1.000.$
2.5,	$+6.875.$
0.0,	$-5.000.$
$-0.5,$	$-5.875.$
$-1.5,$	$-12.125.$

Interpolation (§ 381) shows that if $y = 0$, $x = 1.88+$. Does the result agree with the figure?

450. Consider any rational integral function of x, for example $x^3 + x - \frac{63}{4}$.

Put $$y = x^3 + x - \frac{63}{4}.$$

Assuming values of x, we compute the corresponding values of y, and construct the graph. Now, any value of x which makes $y = 0$ satisfies the equation $x^2 + x - \frac{63}{4} = 0$, and is a root of that equation; hence, any abscissa whose corresponding ordinate is zero represents a root of this equation. The roots may be found, approximately, by measuring the abscissas of the points where the graph meets XX', for at these points $y = 0$.

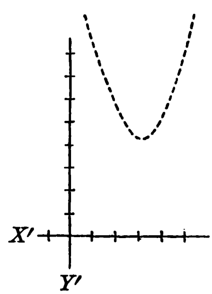


From the given equation the following table may be formed:

If x is	y is	If x is	y is
0,	-15.75.	-1,	-15.75.
1,	-13.75.	-2,	-13.75.
2,	-9.75.	-3,	-9.75.
3,	-3.75.	-4,	-3.75.
4,	$+4.25$.	-5,	$+4.25$.

The table shows that one root is between 3 and 4 (since y changes from $-$ to $+$, and therefore passes through zero); and, for a like reason, the other is between -4 and -5.

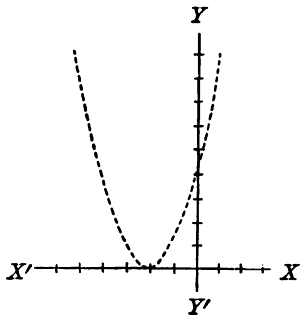
451. An equation of any degree may be thus plotted, and the graph will be found to cross the axis $X'X$ as many times as there are *real* roots in the equation.



When an equation has no real roots, the graph does not meet $X'X$.

In the equation $x^2 - 6x + 13 = 0$, both of whose roots are imaginary, the graph, at its nearest approach, is 4 units distant from $X'X$.

13



If an equation has a double root, its graph touches $X'X$, but does not intersect it.

The equation $x^2+4x+4=0$ has the roots -2 and -2, and the graph is as shown in the figure.

Exercise 71.

Construct the graphs of the following functions:

1. $x^2+3x-10.$
2. $x^3-2x^2+1.$
3. $x^4-20x^2+64.$
4. $x^2-4x+10.$
5. $x^4-5x^2+4.$
6. $x^3-4x^2+x-1.$

DERIVATIVES.

452. Definition. Let x be a variable, and $f(x)$ any function of x.

Suppose x to have a particular value a; the corresponding value of $f(x)$ is $f(a)$ (§ 431).

Now suppose x to increase to $a+h$; the corresponding value of $f(x)$ is $f(a+h)$.

The increase in the value of $f(x)$, called the **increment** of $f(x)$, is $f(a+h)-f(a)$; the increase in the value of x is h.

Dividing the increment of $f(x)$ by the increment of x, we obtain

$$\frac{f(a+h)-f(a)}{h}.$$

In the same manner for any particular value of x; that is, for any value of x.

Hence, in general, we shall obtain the expression

$$\frac{f(x+h)-f(x)}{h},$$

where x may have any value; that is, is variable.

The *limit* which this expression approaches, as h approaches zero as a limit, is called the **derivative** of the function $f(x)$ with respect to x. The derivative of a function of x is, in general, a new function of x.

The above may be written:

Derivative with respect to x of $f(x)$

$$= \underset{h \doteq 0}{\text{limit}} \left[\frac{\text{increment of } f(x)}{\text{increment of } x}\right]$$

$$= \underset{h \doteq 0}{\text{limit}} \left[\frac{f(x+h)-f(x)}{h}\right].$$

NOTE. $h \doteq 0$ is read "as h approaches zero."

The particular value of the derivative corresponding to $x = a$ is

$$\underset{h \doteq 0}{\text{limit}} \left[\frac{f(a+h)-f(a)}{h}\right].$$

An increment may be either positive or negative.

In general, the derivative with respect to u of v, where v is a function of u, is the limit, as the increment of u approaches 0, of

$$\frac{\text{increment of } v}{\text{increment of } u}.$$

The derivative with respect to x of $f(x)$ is represented by $D_x f(x)$; that of $f(y)$ with respect to y by $D_y f(y)$; that of v with respect to u by $D_u v$; and so on.

The derivative of $f(x)$ with respect to x is also represented by $f'(x)$. Thus $D_x f(x) \equiv f'(x)$; $D_y f(y) \equiv f'(y)$; and so on.

453. Rule for finding a Derivative. *In the given function change* $\mathbf{x}$ *to* $\mathbf{x}+\mathbf{h}$.

From the new value of the function subtract the old, and divide the remainder by $\mathbf{h}$.

Take the limit of the quotient as $\mathbf{h}$ *approaches zero as a limit.*

The derivative of a constant is 0, since the increment of a constant will always be 0.

(1) Find $D_x(ax)$.

The function is	ax.
Change x to $x+h$,	$a(x+h)$.
From the new value subtract the old,	ah.
Divide by h,	a.

Take the limit as h approaches 0 as a limit;

$$\therefore D_x(ax)=a.$$

If $a=1$, $\qquad D_x x=1$.

(2) Find $D_x(x^3+4x+1)$.

The function is $\quad x^3+4x+1$.

Change x to $x+h$, $\quad (x+h)^3+4(x+h)+1$,

or $\quad x^3+3hx^2+3h^2x+h^3+4x+4h+1$.

From the new value subtract the old,

$$3hx^2+3h^2x+h^3+4h.$$

Divide by h, $\quad 3x^2+3hx+h^2+4$.

Take the limit as h approaches 0 as a limit;

$$\therefore D_x(x^3+4x+1)=3x^2+4.$$

454. Derivative of x^n. The function is x^n. Changing x to $x+h$, we obtain $(x+h)^n$. Now, whatever the value of n, $(x+h)^n$ can be expanded by the binomial theorem, and we obtain

$$(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{\underline{|2}}x^{n-2}h^2+\cdots\cdots$$

From this new value of the function subtract x^n, the old value, and divide by h.

We now have

$$D_x(x^n) = \underset{h \doteq 0}{\text{limit}}\left[nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2}h + \cdots\cdots\right]$$
$$= nx^{n-1};$$

the sum of the terms after the first approaches 0 as a limit by § 375.

Hence, to find the derivative with respect to x of any power of x, *multiply by the exponent, and diminish the exponent of x by one.*

Thus,
$$D_x(x^4) = 4x^3; \quad D_x(x^{-3}) = -3x^{-4};$$
$$D_x\frac{1}{\sqrt{x^3}} = D_x(x^{-\frac{3}{2}}) = -\tfrac{3}{2}x^{-\frac{5}{2}}.$$

Exercise 72.

Find the derivatives with respect to x of:

1. x^2.
2. x^3.
3. $\frac{1}{x}$.
4. x^{-2}.
5. x^4.
6. $\frac{1}{x^3}$.
7. x^{-4}.
8. $x^2 + x$.
9. $x^3 + 2x^2$.
10. $(x+a)^2$.
11. $\frac{1}{x^2-3}$.
12. $(x+1)^{-2}$.

---

455. Derivative of a Sum. Let $f(x)$ and $\phi(x)$ be two functions of x; their sum $f(x) + \phi(x)$ is also a function of x.

Now,

$$D_x[f(x) + \phi(x)] = \underset{h \doteq 0}{\text{limit}}\left[\frac{f(x+h) + \phi(x+h) - f(x) - \phi(x)}{h}\right]$$
$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{f(x+h) - f(x)}{h}\right] + \underset{h \doteq 0}{\text{limit}}\left[\frac{\phi(x+h) - \phi(x)}{h}\right]$$
$$= D_x f(x) + D_x \phi(x).$$

Similarly for the sum of any number of functions.

The above may be formulated,

$$D_x(f + \phi + \cdots\cdots) = D_x f + D_x \phi + \cdots\cdots$$

Here f is an abbreviation for $f(x)$, ϕ for $\phi(x)$, etc.

By means of the above and §§ 453, 454, we can find t derivative with respect to x of any rational integral fur tion of x.

Ex. Find $D_x(2x^3 + 4x^2 - 8x + 3)$.

$$\begin{aligned} D_x(2x^3 + 4x^2 - 8x + 3) &= D_x(2x^3) + D_x(4x^2) - D_x(8x) + D_x(3) \\ &= 2D_x x^3 + 4D_x x^2 - 8D_x x + D_x 3 \\ &= 2(3x^2) + 4(2x) - 8(1) + 0 \\ &= 6x^2 + 8x - 8. \end{aligned}$$

456. Derivative of a Product. Let $f(x)$ and $\phi(x)$ be tw functions of x; their product $f(x)\phi(x)$ is a new functio of x.

Now,

$$D_x[f(x)\phi(x)] = \underset{h \doteq 0}{\text{limit}}\left[\frac{f(x+h)\phi(x+h) - f(x)\phi(x)}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{f(x+h)\phi(x+h) - f(x+h)\phi(x) + f(x+h)\phi(x) - f(x)\phi(x)}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[f(x+h)\frac{\phi(x+h) - \phi(x)}{h}\right] + \underset{h \doteq 0}{\text{limit}}\left[\phi(x)\frac{f(x+h) - f(x)}{h}\right]$$

$$= f(x)D_x\phi(x) + \phi(x)D_x f(x),$$

since $$\underset{h \doteq 0}{\text{limit}}[f(x+h)] = f(x).$$

The above may be formulated

$$D_x(f\phi) = fD_x\phi + \phi D_x f.$$

Similarly for three or more functions. Thus,

$$D_x(f\phi F) = f\phi D_x F + fFD_x\phi + \phi FD_x f.$$

457. Derivative of $(x-a)^n$.

$$D_x(x-a)^n = \underset{h \doteq 0}{\text{limit}}\left[\frac{(x-a+h)^n - (x-a)^n}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[\frac{(x-a)^n + n(x-a)^{n-1}h + \cdots\cdots - (x-a)^n}{h}\right]$$

$$= \underset{h \doteq 0}{\text{limit}}\left[n(x-a)^{n-1} + \cdots\cdots\right]$$

$$= n(x-a)^{n-1}.$$

Ex. $$D_x(x-3)^4 = 4(x-3)^3.$$

Exercise 73.

Write the derivatives with respect to x of:

1. $x^2 + 4$.
2. $x^3 + 3x^2 - 1$.
3. $x^4 + x^2 + 2$.
4. $x^5 - 3x^4 + x^3$.
5. $4x^4 + 6x^3 + 2$.
6. $6x^5 - 7x^2 + 7x$.
7. $3x^5 + 4x^4 + x^3 - x^2 - 6x + 5$.
8. $4x^5 - 2x^4 - x^3 + 6x^2 - 7$.
9. $(x-2)(x+3)$.
10. $(x-1)(x-2)(x-3)$.
11. $(x-3)^2(x+4)$.
12. $(x-4)^2(x-2)(x+1)$.
13. $(x-\alpha)^2(x-\beta)^2$.
14. $(x-\alpha)(x-\beta)(x-\gamma)$.
15. $(x-2)(x-3)(x+5)(x+4)$.
16. $(x^2+2)(x^2-4x+8)$.

458. Successive Derivatives. The derivative of a function of x is itself a function of x, and has itself a derivative with respect to x.

The derivative of the derivative is called the **second derivative**; the derivative of the second derivative, the **third derivative**; and so on.

By *derivative* is meant the first derivative, unless the contrary is expressly stated.

The second derivative with respect to x of $f(x)$ is represented by $D_x^2 f(x)$, or by $f''(x)$; the third derivative by $D_x^3 f(x)$, or by $f'''(x)$; and so on.

Evidently, $f''(x) = D_x^2 f(x) = D_x D_x f(x)$;

$$f'''(x) = D_x^3 f(x) = D_x D_x^2 f(x) = D_x D_x D_x fx;$$

and so on.

459. Values of the Derivatives. The value which $f(x)$ assumes when for x we put a is represented by $f(a)$.

Similarly, the value which $f'(x)$ assumes when for x we put a is represented by $f'(a)$; the value of $f''(x)$ by $f''(a)$; and so on.

Thus, if $\qquad f(x) \equiv x^3 - 2x^2 + x + 4,$

we obtain $\qquad f'(x) \equiv 3x^2 - 4x + 1,$

$$f''(x) \equiv 6x - 4,$$

$$f'''(x) \equiv 6,$$

$f^{\text{iv}}(x), f^{\text{v}}(x)$, etc., all vanish.

Putting 2 for x we obtain

$$f(2) = 6; \; f'(2) = 5; \; f''(2) = 8; \; f'''(2) = 6.$$

Similarly for any function.

460. Sign of the Derivative. In the function $f(x)$ let x increase by the successive addition of very small increments. As x increases, the value of $f(x)$ will change, sometimes increasing, sometimes decreasing.

Suppose x to have reached a fixed value a; the corresponding values of $f(x)$ and $f'(x)$ will be $f(a)$ and $f'(a)$.

Let x increase by an increment h from a to $a+h$.

By § 452, $$f'(a) = \underset{h \doteq 0}{\text{limit}}\left[\frac{f(a+h)-f(a)}{h}\right].$$

If $f(x)$ is *increasing* as x passes through the value a, $f(a+h) > f(a)$ and $f'(a)$ is *positive*.

If $f(x)$ is *decreasing* as x passes through the value a, $f(a+h) < f(a)$ and $f'(a)$ is *negative*.

Conversely, if $f'(a)$ is positive, $f(a+h)-f(a)$ is positive, and $f(x)$ is increasing as x passes through the value a.

If $f'(a)$ is negative, $f(a+h)-f(a)$ is negative, and $f(x)$ is decreasing as x passes through the value a.

Hence, for a particular value of x, if $f'(x)$ is positive, $f(x)$ is increasing; and if $f'(x)$ is negative, $f(x)$ is decreasing. And conversely.

Observe that we are speaking of increasing and decreasing *algebraically*.

Ex. Take the function

$$f(x) \equiv x^3 - 3x^2 - 6x + 10.$$

Here $$f'(x) \equiv 3x^2 - 6x - 6.$$

We find $$f(1) = 2,\ f'(1) = -9.$$

$\therefore f(x)$ is decreasing as x passes through the value 1; for example,

$$f(1) = 2,\ f(1.1) = 1.101, \text{ and } 1.101 < 2.$$

Again, $$f(3) = -8,\ f'(3) = +3.$$

$\therefore f(x)$ is increasing as x passes through the value 3; for example,

$$f(3) = -8,\ f(3.1) = -7.639, \text{ and } -7.639 > -8.$$

Exercise 74.

Write the successive derivatives with respect to x of:

1. $x^3 - 4x^2 + 2$.
2. $x^3 + 4x^2 - 5x$.
3. $2x^3 + 2x^2 - 4x + 1$.
4. $3x^4 + 3x^3 - x^2 + x$.
5. $ax^3 + 3bx^2 + 3cx + d$.
6. $ax^4 + 4bx^3 + 6cx^2 + 4dx + e$.
7. $(x-\alpha)^2(x-\beta)$.
8. $(x-\alpha)(x-\beta)(x-\gamma)$.
9. $(x-\alpha)^2(x-\beta)^2$.

Find whether the following functions are increasing or decreasing as x increases through the value set opposite each of them:

10. $x^3 - x^2 + 1$ 2.
11. $x^4 - x^2 + 6x - 1$ 4.
12. $2x^4 + 3x^2 - 6x$ 1.
13. $4x^4 - 3x^2 + 4x - 6$ -3.

---

461. Derivative in Terms of the Roots. Take the cubic

$$f(x) \equiv a(x-\alpha)(x-\beta)(x-\gamma),$$

since $D_x(x-\alpha)=1$, $D_x(x-\beta)=1$, $D_x(x-\gamma)=1$, (§ 457) we have, by § 456,

$$f'(x) \equiv a(x-\beta)(x-\gamma) + a(x-\alpha)(x-\gamma) + a(x-\alpha)(x-\beta)$$

$$\equiv \frac{f(x)}{x-\alpha} + \frac{f(x)}{x-\beta} + \frac{f(x)}{x-\gamma}.$$

Similarly, for any quantic,

$$f'(x) \equiv \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \cdots\cdots \frac{f(x)}{x-\alpha_n} \equiv \sum \frac{f(x)}{x-\alpha}.$$

462. Multiple Roots. In the quantic $f(x)$ let a be a triple root. Then we can write (§ 439),

$$f(x) \equiv (x-a)^3 \phi(x)$$

where the degree of $\phi(x)$ is less by 3 than that of $f(x)$.

By § 456, $f'(x) \equiv (x-a)^3 \phi'(x) + 3(x-a)^2 \phi x$
$$\equiv (x-a)^2[(x-a)\phi' x + 3\phi x].$$

Hence, if $f(x)$ has a triple root a, the factor $(x-a)^2$ occurs in the H.C.F. of $f(x)$ and $f'(x)$.

Similarly for a multiple root of any order.

To find the multiple roots of $f(x)$.

Find the H.C.F. of $f(x)$ and $f'(x)$, and resolve it into factors. Each root will occur once more in $f(x)$ than the corresponding factor occurs in the H.C.F.

Ex. Find the multiple roots of

$$x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0.$$

Here $\quad f(x) \equiv x^5 - x^4 - 5x^3 + x^2 + 8x + 4.$

$\qquad\quad f'(x) \equiv 5x^4 - 4x^3 - 15x^2 + 2x + 8.$

Find the H.C.F. of $f(x)$ and $f'(x)$ as follows:

$5-4-15+2+8$	$5-5-25+5+40+20$	$1-1$
$5+0-15-10$	$5-4-15+2+8$	
$-4+0+12+8$	$-1-10+3+32+20$	
$-4+0+12+8$	$-5-50+15+160+100$	
	$-5+4+15-2-8$	
	$54)-54+0+162+108$	
	$-1+0+3+2$	$-5+4$

Hence, $x^3 - 3x - 2$ is the H.C.F.

We find, by trial, that -1 is a root of the equation

$$x^3 - 3x - 2 = 0.$$

The other roots are found to be -1 and 2 (§ 441).

Hence, $\quad x^3 - 3x - 2 \equiv (x+1)^2(x-2).$

Therefore, -1 is a triple root, and 2 is a double root, of the given equation. As the given equation is of the fifth degree, these are all the roots, and the equation may be written

$$(x+1)^3(x-2)^2=0.$$

Having found the multiple roots of an equation, we may divide by the corresponding factors, and find the remaining roots, if any, from the reduced equation.

Exercise 75.

The following equations have multiple roots. Find all the roots of each equation:

1. $x^3-8x^2+13x-6=0.$
2. $x^3-7x^2+16x-12=0.$
3. $x^4-6x^2-8x-3=0.$
4. $x^4-7x^3+9x^2+27x-54=0.$
5. $x^4+6x^3+x^2-24x+16=0.$
6. $x^5-11x^4+19x^3+115x^2-200x-500=0.$
7. Resolve into linear factors

$$x^6-5x^5+5x^4+9x^3-14x^2-4x+8.$$

8. Show that an equation of the form $x^n=a^n$ can have no multiple root.

9. Show that the condition that the equation

$$x^3+3qx+r=0$$

shall have a double root is $4q^3+r^2=0.$

10. Show that the condition that the equation

$$x^3+3px^2+r=0$$

shall have a double root is $r(r-2p^3)=0.$

463. Expansion of f(x + h). Consider a quantic of the fourth degree, viz.:

$$f(x) \equiv ax^4 + bx^3 + cx^2 + dx + e.$$

Put $x + h$ in place of x, then

$$f(x+h) \equiv a(x+h)^4 + b(x+h)^3 + c(x+h)^2 \\ + d(x+h) + e.$$

Expanding the powers of $x + h$, and arranging the terms by descending powers of x, the above identity becomes

$$f(x+h) \equiv a \left| \begin{array}{r} x^4 + 4ah \\ +\ b \\ \\ \\ \\ \end{array} \right| \begin{array}{r} x^3 + 6ah^2 \\ +3bh \\ +\ c \\ \\ \\ \end{array} \left| \begin{array}{r} x^2 + 4ah^3 \\ +3bh^2 \\ +2ch \\ +\ d \\ \\ \end{array} \right| \begin{array}{r} x + ah^4 \\ +bh^3 \\ +ch^2 \\ +dh \\ +e \end{array}$$

But
$$\begin{aligned} f(h) &= ah^4 + bh^3 + ch^2 + dh + e, \\ f'(h) &= 4ah^3 + 3bh^2 + 2ch + d \\ f''(h) &= 12ah^2 + 6bh + 2c \\ f'''(h) &= 24ah + 6b \\ f^{\text{iv}}(h) &= 24a, \\ f^{\text{v}}(h) &= 0, \end{aligned}$$

and we have

$$f(x+h) \equiv f(h) + f'(h) + x^2\frac{f''(h)}{\underline{|2}} + x^3\frac{f'''(h)}{\underline{|3}} + x^4\frac{f^{\text{iv}}(h)}{\underline{|4}}.$$

3

If we had arranged the expansion of $f(x+h)$ by powers of h, we should have found

$$f(x+h) \equiv f(x) + hf'(x) + h^2\frac{f''(x)}{\underline{|2}} + h^3\frac{f'''(x)}{\underline{|3}} + h^4\frac{f^{\text{iv}}(x)}{\underline{|4}}.$$

Similarly for any quantic.

464. Calculation of the Coefficients. The coefficients in the expansion of $f(x+h)$ may be conveniently calculated as follows:

Take $\quad f(x) \equiv ax^4 + bx^3 + cx^2 + dx + e.$

Put $\quad f(x+h) \equiv Ax^4 + Bx^3 + Cx^2 + Dx + E,$

where A, B, C, D, E are to be found.

In the last identity put $x - h$ for x.

Then, since $f(x - h + h) = f(x)$, we obtain

$$f(x) \equiv A(x-h)^4 + B(x-h)^3 + C(x-h)^2 + D(x-h) + E.$$

From the last identity we derive the following rule for finding the coefficients of the powers of x in the expansion of $f(x+h)$.

Divide $f(x)$ by $x - h$; the remainder will be E, that is $f(h)$; and the quotient

$$A(x-h)^3 + B(x-h)^2 + C(x-h) + D.$$

Divide this quotient by $(x - h)$; the remainder will be D, that is $f'(h)$; and the quotient

$$A(x-h)^2 + B(x-h) + C.$$

Continuing the division the last quotient will be A or a.

The above division is best arranged as follows (§ 436):

$$\begin{array}{lllll|l}
a & b & c & d & e & \underline{h} \\
 & ah & b'h & c'h & d'h & \\
\hline
a & b' & c' & d' & E & \\
 & ah & b''h & c''h & & \\
\hline
a & b'' & c'' & D & & \\
 & ah & b'''h & & & \\
\hline
a & b''' & C & & & \\
 & ah & & & & \\
\hline
a & B & & & &
\end{array}$$

and we have

$$f(x+h) \equiv ax^4 + Bx^3 + Cx^2 + Dx + E.$$

The above method is easily extended to equations of any degree.

Exercise 76.

In the following quantics put for x the expression opposite, and reduce.

1. $x^3 - 3x^2 + 4x - 6$ $\qquad x + 2.$
2. $x^4 - 2x^2 + 6x - 3$ $\qquad x + 4.$
3. $3x^4 - 2x^3 + 2x^2 - x - 4$ $\qquad x + 3.$
4. $2x^4 - 3x^3 + 6x^2 - 7x - 8$ $\qquad x - 2.$
5. $2x^4 - 2x^3 + 4x^2 - 5x - 4$ $\qquad x - 3.$

TRANSFORMATION OF EQUATIONS.

465. The solution of an equation, and the investigation of its properties, is often facilitated by a change in the form of the equation. Such a change of form is called a **transformation** of the equation.

466. Roots with Signs changed. *The roots of the equation* $\mathbf{f}(-\mathbf{x}) = 0$ *are those of the equation* $\mathbf{f}(\mathbf{x}) = 0$, *each with its sign changed.*

For, let a be any root of equation $f(x) = 0$.

Then, we must have $f(a) = 0$.

In the quantic $f(-x)$ put $-a$ for x; that is, a for $-x$.

The result is $f(a)$.

But we have just seen that $f(a)$ vanishes, since a is a root of the equation $f(x) = 0$. Hence, $f(-x)$ vanishes when we put $-a$ for x, and (§ 432) $-a$ is therefore a root of the equation $f(-x) = 0$.

Similarly, the negative of each of the roots of $f(x)=0$ is a root of $f(-x)=0$; and, since the two equations are evidently of the same degree, these are all the roots of the equation $f(-x)=0$.

To obtain the $f(-x)$ we change the sign of all the odd powers of x in the quantic $f(x)$.

Thus, the roots of the equation

$$x^4-2x^3-13x^2+14x+24=0$$

are 2, 4, -1, -3; and those of the equation

$$x^4+2x^3-13x^2-14x+24=0$$

are -2, -4, $+1$, $+3$.

467. Roots multiplied by a Given Number. Consider the equation

$$ax^4+bx^3+cx^2+dx+e=0. \quad (1)$$

Put $y=mx$, then $x=\dfrac{y}{m}$; and the equation becomes

$$a\left(\frac{y}{m}\right)^4+b\left(\frac{y}{m}\right)^3+c\left(\frac{y}{m}\right)^2+d\left(\frac{y}{m}\right)+e=0. \quad (2)$$

The left-member of (2) differs from the left-member of (1) only in that $\dfrac{y}{m}$ is put in place of x.

Let α be any root of (1); the left-member of (1) vanishes when we put α for x, and we obtain

$$a\alpha^4+b\alpha^3+c\alpha^2+d\alpha+e=0.$$

In the left-member of (2) put $m\alpha$ for y; we obtain

$$a\alpha^4+b\alpha^3+c\alpha^2+d\alpha+e,$$

which, as we have just seen, vanishes. Hence, if α is a root of (1), $m\alpha$ is a root of (2). Since the above is true for each of the roots of (1), and the two equations are evidently of the same degree, the roots thus obtained are all the roots of (2).

Similarly, for an equation of any degree.

Equation (2) may be written in the form

$$ay^4 + mby^3 + m^2cy^2 + m^3dy + m^4e = 0.$$

The above form, if written with x in place of y, gives the following rule:

Multiply the second term by m; the third term by m^2; and so on. Zero coefficients are to be supplied for missing powers of x.

Ex. Write the equation of which the roots are the doubles of the roots of the equation

$$3x^4 - 2x^3 + 4x^2 - 6x - 5 = 0.$$

Here $m = 2$, and the result is

$$3x^4 - 2(2)x^3 + 4(2)^2x^2 - 6(2)^3x - 5(2^4) = 0,$$

or

$$3x^4 - 4x^3 + 16x^2 - 48x - 80 = 0.$$

468. Removal of Fractional Coefficients. If any of the coefficients of an equation in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots\cdots p_n = 0$$

are fractions, we can remove fractions as follows:

Multiply the roots by m; then take m so that all of the coefficients will be integers.

Ex. Reduce to an equation, in the p form, with integral coefficients

$$2x^3 - \tfrac{1}{3}x^2 + \tfrac{5}{6}x + \tfrac{1}{4} = 0.$$

Dividing by 2, $\quad x^3 - \frac{1}{6}x^2 + \frac{5}{12}x + \frac{1}{8} = 0.$

Multiplying the roots by m (§ 467),

$$x^3 - \frac{m}{6}x^2 + \frac{5m^2}{12}x + \frac{m^3}{8} = 0.$$

The least value of m that will render the coefficients all integral is seen to be 6. Putting 6 for m, we obtain

$$x^3 - x^2 + 15x + 27 = 0,$$

the equation required.

Any multiple of 6 might have been used instead of 6, but the smaller the number the easier the work.

469. Reciprocal Roots. Consider the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0. \qquad (1)$$

Put $y = \frac{1}{x}$; then $x = \frac{1}{y}$; and the equation becomes

$$a\left(\frac{1}{y}\right)^4 + b\left(\frac{1}{y}\right)^3 + c\left(\frac{1}{y}\right)^2 + d\left(\frac{1}{y}\right) + e = 0. \qquad (2)$$

The left-member of (2) differs from the left-member of (1) only in that $\frac{1}{y}$ is put in place of x.

Let a be any root of (1); then we must have

$$aa^4 + ba^3 + ca^2 + da + e = 0.$$

In the left-member of (2) put a for $\frac{1}{y}$; that is, $\frac{1}{a}$ for y; we obtain

$$aa^4 + ba^3 + ca^2 + da + e,$$

which, as we have just seen, vanishes.

Hence, $\frac{1}{a}$ is a root of (2). Since the above is true for each of the roots of (1), and the two equations are evidently of the same degree, the reciprocals of the roots of (1) are all the roots of (2).

Similarly for an equation of any degree.

Equation (2) may be written

$$a + by + cy^2 + dy^3 + ey^4 = 0,$$

or, writing x in place of y,

$$ex^4 + dx^3 + cx^2 + bx + a = 0;$$

so that the coefficients are those of the given equation in reversed order.

Ex. Write the equation of which the roots are the reciprocals of the roots of $\quad 2x^4 - 3x^3 + 4x^2 - 5x - 7 = 0.$

The result is $\quad 2 - 3x + 4x^2 - 5x^3 - 7x^4 = 0,$

or $\quad 7x^4 + 5x^3 - 4x^2 + 3x - 2 = 0.$

470. Reciprocal Equations. The coefficients of an equation may be such that reversing their order does not change the equation. In this case the reciprocal of a root is another root of the equation. That is, one-half the roots are reciprocals of the other half.

An equation in which the above is true is called a **reciprocal equation.**

Thus, the roots of the equation

$$6x^5-29x^4+27x^3+27x^2-29x+6=0$$

are $-1, 2, 3, \frac{1}{2}, \frac{1}{3}$. Here, -1 is the reciprocal of itself; $\frac{1}{2}$ of 2; $\frac{1}{3}$ of 3.

471. Roots diminished by a Given Number. Consider the equation

$$ax^4+bx^3+cx^2+dx+e=0. \quad (1)$$

To obtain the equation which has for its roots the roots of the above equation each diminished by h, we proceed as follows:

Put $y=x-h$; then $x=y+h$; and the equation becomes

$$a(y+h)^4+b(y+h)^3+c(y+h)^2+d(y+h)+e=0. \quad (2)$$

The left-member of (2) differs from the left-member of (1) only in that $y+h$ is put in place of x.

Let a be any root of (1); then we must have

$$aa^4+ba^3+ca^2+da+e=0.$$

In the left-member of (2) put a for $y+h$; that is, $a-h$ for y; we obtain

$$aa^4+ba^3+ca^2+da+e;$$

which, as we have just seen, vanishes.

Hence, $a-h$ is a root of (2). Since the above is true for each of the roots of (1), and the two equations are evidently of the same degree, the roots thus found are all the roots of equation (2).

Similarly for an equation of any degree.

3

Putting x in place of y in equation (2), that equation may be written $f(x+h)=0$, equation (1) being $f(x)=0$.

Equation (2) may be also written (§ 463) in the form

$$ax^4+Bx^3+Cx^2+Dx+E=0,$$

where $E=f(h)$, $D=f'(h)$, $C=\dfrac{f''(h)}{\underline{|2}}$, $B=\dfrac{f'''(h)}{\underline{|3}}$.

The coefficients are most easily calculated by the method explained in § 464.

To *increase* the roots by a given number h, we *diminish* the roots by $-h$.

Ex. Obtain the equation which has for its roots the roots of the equation

$$2x^4-3x^3-4x^2+2x+9=0,$$

each diminished by 2.

The work (§ 464) is as follows:

2	-3	-4	$+2$	$+9$	$\underline{\lfloor 2}$
	$+4$	$+2$	-4	-4	
2	$+1$	-2	-2	$+5$	
	$+4$	$+10$	$+16$		
2	$+5$	$+8$	$+14$		
	$+4$	$+18$			
2	$+9$	$+26$			
	$+4$				
2	$+13$				

The required equation is

$$2x^4+13x^3+26x^2+14x+5=0.$$

472. Transformation in General. In the general problem of transformation we have given an equation in x, as $f(x)=0$, and we have to form a new equation in y where y is a given function of x, such as $\phi(x)$.

When from the equation $y=\phi(x)$ we can find an expression for x, the transformation can be readily accomplished

by substituting this expression for x in the given equation, and reducing the result.

(1) Given the equation

$$x^3 - 3x + 1 = 0,$$

to find the equation in y where $y = 3x - 2$.

We find $x = \frac{y+2}{3}$. Substituting this expression for x in the given equation, that becomes

$$\left(\frac{y+2}{3}\right)^3 - 3\left(\frac{y+2}{3}\right) + 1 = 0,$$

which reduces to

$$y^3 + 6y^2 - 15y - 19 = 0.$$

(2) Given the equation

$$x \; - 2x^2 + 3x - 5 = 0,$$

of which α, β, γ are the roots. Find the equation of which the roots are

$$\beta + \gamma - \alpha, \;\; \gamma + \alpha - \beta, \;\; \alpha + \beta - \gamma.$$

We have

$$\begin{aligned} y &= \beta + \gamma - \alpha \\ &= \alpha + \beta + \gamma - 2\alpha \\ &= 2 - 2\alpha. \end{aligned}$$ § 442

$$\therefore \; \alpha = \frac{2-y}{2}.$$

But, since α is a root of the given equation,

$$\alpha^3 - 2\alpha^2 + 3\alpha - 5 = 0.$$

Putting $\frac{2-y}{2}$ for α, and reducing, we obtain

$$y^3 - 2y^2 + 8y + 24 = 0,$$

the equation required.

Exercise 77.

Multiply the roots of each of the following equations by the number placed opposite the equation.

1. $x^3 - 3x^2 + 2x - 4 = 0$ $\qquad -1.$
2. $x^4 + 3x^2 - 2x - 1 = 0$ $\qquad -2.$
3. $2x^4 - 3x^3 + x^2 - 6x - 4 = 0$ $\qquad -3.$
4. $2x^4 - 3x^2 + 6x - 8 = 0$ $\qquad -2.$
5. $3x^5 - 4x^3 - 2x + 7 = 0$ $\qquad -2.$

Transform to equations with integral coefficients in the p form the equations

6. $12x^3 - 4x^2 + 6x + 1 = 0.$
7. $6x^3 + 10x^2 - 7x + 16 = 0.$
8. $10x^4 + 5x^3 - 4x^2 + 25x - 30 = 0.$
9. $6x^5 + 3x^4 + 4x^3 - 2x^2 + 6x - 18 = 0.$

Write the equations which have for their roots the reciprocals of the roots of the following equations:

10. $3x^4 - 2x^3 + 5x^2 - 6x + 7 = 0.$
11. $2x^5 - 4x^3 - 5x^2 - 7x - 8 = 0.$
12. $x^6 - x^4 + 2x^2 + 4x - 1 = 0.$

Diminish the roots of each of the following equations by the number opposite the equation:

13. $x^3 - 11x^2 + 31x - 12 = 0$ $\qquad 1.$
14. $x^4 - 6x^3 + 4x^2 + 18x - 5 = 0$ $\qquad 2.$
15. $x^3 + 10x^2 + 13x - 24 = 0$ $\qquad -2.$

16. $x^4+x-16x^2-4x+48=0$ 4.

17. $x^4+x^2-3x+4=0$ 0.3.

18. $x^4-3x^3-x^2+4x-5=0$ -0.4.

19. Form the equation which has for its roots the squares of the roots of the equation

$$x^3-2x^2+3x-5=0.$$

20. Form the equation which has for its roots the squares of the differences of the roots of

$$x^3-4x^2+2x-3=0.$$

21. Given the equation

$$x^3-2x^2+4x-4;$$

find the equation in y where $y=2x^2-3$.

SITUATION OF THE ROOTS.

473. Finite Value of a Quantic. Any positive integral power of x is finite as long as x is finite.

The product of a positive integral power of x by a finite number will be finite when x is finite.

A quantic consists of the sum of a definite number of such products, and will, consequently, have a finite value as long as x is finite.

The *derivatives* of a quantic are new quantics, and will, consequently, have finite values as long as x is finite.

474. Sign of a Quantic. *When* **x** *is taken numerically large enough, the sign of a quantic is the same as the sign of its first term.*

Write the quantic

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots\cdots a_n$$

in the form $$a_0x^n\left(1+\frac{a_1}{a_0x}+\frac{a_2}{a_0x^2}+\cdots\cdots\frac{a_n}{a_0x^n}\right).$$

By taking x large enough, each of the terms in parentheses after the first can be made as small as we please.

If a_k is numerically the greatest of the coefficients $a_1, a_2, \ldots\ldots a_n$, the sum of the terms in parenthesis after the first will be numerically less than

$$\frac{a_k}{a_0}\left(\frac{1}{x}+\frac{1}{x^2}+\cdots\cdots\frac{1}{x^n}\right);$$

that is (§ 231), less than $\frac{a_k}{a_0}\left(\frac{1-\frac{1}{x^n}}{x-1}\right)$.

The value of this expression can be made less than 1, or indeed less than *any* assigned value, by taking x large enough.

Hence, even in the most unfavorable case, that in which all the terms in parenthesis after the first are negative, the sum of these terms can still be made less than 1; the sum of all the terms in parenthesis will then be positive. The sign of the quantic will be the same as the sign of a_0x^n, its first term.

475. *When* x *is taken numerically small enough, the sign of a quantic is the same as the sign of its last term.*

Write the quantic in the form

$$a_n\left(\frac{a_0x^n}{a^n}+\frac{a_1x^{n-1}}{a^n}+\cdots\cdots\frac{a_{n-1}x}{a^n}+1\right).$$

The proof follows the method of the last section.

476. Continuity of a Rational Integral Function. A function of x, $f(x)$, is **continuous** when an infinitesimal (§ 361) change in x always produces an infinitesimal change in $f(x)$, *whatever the value of x.*

We proceed to show that if $f(x)$ is a rational integral function of x, it is a continuous function.

Give to x *any* particular finite value a; the corresponding value of $f(x)$ is $f(a)$.

Increase x to $a+h$; the corresponding value of $f(x)$ is $f(a+h)$, and the increment in the value of $f(x)$ is

$$f(a+h)-f(a),$$

or (§ 363),

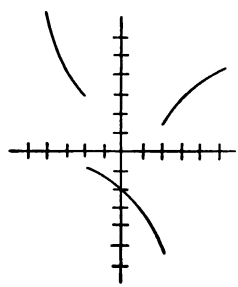
$$h\left(f'(a)+\frac{h}{\lfloor 2}f''(a)+\cdots\cdots\frac{h^{n-1}}{\lfloor n}f^{n}(a)\right).$$

The derivatives $f'(a)$, $f''(a)$, ... $f^{n}(a)$ all have finite values (§ 473); and it is easily seen from § 475 that when h is very small the expression in parenthesis is numerically less than $2f'(a)$. Since $2hf'(a)$ approaches 0 as a limit (§ 363, I.) when h approaches 0 as a limit, the increment of $f(x)$, which is less than $2hf'(a)$, will approach 0 as a limit when h approaches 0 as a limit.

Since the above is true for *any particular finite* value of x, we see that an infinitesimal change in x always produces an infinitesimal change in $f(x)$.

It follows that as $f(x)$ gradually changes from $f(a)$ to $f(b)$, it must pass through all intermediate values.

The derivatives of a rational integral function of x are themselves rational integral functions of x, and are therefore continuous.



The changes in the value of a quantic $f(x)$ are well illustrated by the graph of the function. Since $f(x)$ is continuous, we can never have a graph in which there are *breaks* in the curve, as in the curve here given. In this curve there are breaks, or *discontinuities*, at $x=-2$, and $x=+2$.

477. Theorem on Change of Sign. *Let two real numbers* **a** *and* **b** *be put for* **x** *in* **f(x)**. *If the resulting values of* **f(x)** *have contrary signs, an odd number of roots of the equation* **f(x) = 0** *lie between* **a** *and* **b**.

As x changes from a to b, passing through all intermediate values, $f(x)$ will change from $f(a)$ to $f(b)$, passing through all intermediate values. Now, in changing from $f(a)$ to $f(b)$, $f(x)$ changes sign.

Hence, $f(x)$ must pass through the value zero. That is, there is some value of x between a and b which causes $f(x)$ to vanish; that is, some root of the equation $f(x) = 0$ lies between a and b.

But $f(x)$ may pass through zero more than once. To change sign, $f(x)$ must pass through zero an *odd* number of times; and an odd number of roots must lie between a and b.

Applied to the graph of the equation, since to a root corresponds a point in which the graph meets the axis of x (§ 450), the above simply means that to pass from a point below the axis of x to a point above that axis, we must cross the axis an odd number of times.

Thus, in $$x^3 - 2x^2 + 3x - 7 = 0.$$

If we put 2 for x, the value of the left-member is -1; if we put 3 for x, the value is $+11$. Hence, certainly one root, and possibly three roots, of the equation lie between 2 and 3.

478. *An equation of odd degree has at least one real root.*

For, if the first coefficient is not positive, change signs so as to make it positive. If the last term is negative, make x positive and very large; the sign of the left-member is $+$ (§ 474). Put $x = 0$; the sign of the left-member is $-$. Hence, there is at least one real positive root.

Similarly, if the last term is positive, there is at least one real negative root.

479. Descartes' Rule of Signs. An equation in which all the powers of x from x^0 to x^n are present is said to be **complete**; if any powers of x are missing, the equation is said to be **incomplete.** An incomplete equation can be made complete by writing the missing powers of x with zero coefficients.

A **permanence** of sign occurs when $+$ follows $+$, or $-$ follows $-$; a **variation** of sign when $-$ follows $+$, or $+$ follows $-$.

Thus, in the complete equation

$$x^6 - 3x^5 + 2x^4 + x^3 - 2x^2 - x - 3,$$

writing only the signs

$$+ \quad - \quad + \quad + \quad - \quad - \quad -,$$

we see that there are three variations of sign and three permanences.

For *positive* roots, Descartes' rule is as follows:

The number of positive roots of the equation $\mathbf{f}(\mathbf{x}) = 0$ *cannot exceed the number of variations of sign in the quantic* $\mathbf{f}(\mathbf{x})$.

To prove this it is only necessary to prove that for every positive root introduced into an equation there is one variation of sign added.

Suppose the signs of a quantic to be

$$+ \quad - \quad + \quad + \quad + \quad - \quad - \quad +,$$

and introduce a new positive root. We multiply by $x - h$. or, writing only the signs, by $+ \,-$. The result is

$$\begin{array}{ccccccccc}
+ & - & + & + & + & - & - & + & \\
+ & - & & & & & & & \\
\hline
+ & - & + & + & + & - & - & + & \\
 & - & + & - & - & - & + & + & - \\
\hline
+ & - & + & \pm & \pm & - & \mp & + & -
\end{array}$$

The ambiguous signs $\pm$, $\mp$ indicate that there is doubt whether the term is positive or negative. Examining the product we see that to permanences in the multiplicand correspond ambiguities in the product. Hence, we cannot have a greater number of permanences in the product than in the multiplicand, and many have a less number. But there is one more term in the product than in the multiplicand. Hence we have *at least* one more *variation* in the product than in the multiplicand.

For each positive root introduced we have at least one more variation of sign. Hence the number of positive roots cannot exceed the number of variations of sign.

Negative Roots. Change x to $-x$. The negative roots of the given equation will be positive roots of this latter equation (§ 466), and the preceding rule may then be applied.

480. From Descartes' rule we obtain the following:

If the signs of the terms of an equation are all positive the equation has no positive root.

If the signs of the terms of a complete equation are alternately positive and negative, the equation has no negative root.

If the roots of a complete equation are all real, the number of positive roots is the same as the number of variations of sign, and the number of negative roots is the same as the number of permanences of sign.

481. Existence of Imaginary Roots. In an incomplete equation Descartes' rule sometimes enables us to detect the presence of imaginary roots.

Thus, the equation $$x^3 + 5x + 7 = 0$$
may be written $$x^3 \pm 0x^2 + 5x + 7 = 0.$$

We are at liberty to assume that the second term is positive, or that it is negative.

Taking it positive, we have the signs

$$+ \quad + \quad + \quad +;$$

there is no variation, and the equation has no positive root.

Taking it negative, we have the signs

$$+ \quad - \quad + \quad +;$$

there is but one permanence, and therefore not more than one negative root.

As there are three roots, and as imaginary roots enter in pairs, the given equation has one real negative root and two imaginary roots.

Exercise 78.

All the roots of the equations given below are real; determine their signs.

1. $x^4 + 4x^3 - 43x^2 - 58x + 240 = 0.$

2. $x^3 - 22x^2 + 155x - 350 = 0.$

3. $x^4 + 4x^3 - 35x^2 - 78x + 360 = 0.$

4. $x^3 - 12x^2 - 43x - 30 = 0.$

5. $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0.$

6. $x^3 - 12x^2 + 47x - 60 = 0.$

7. $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0.$

8. $x^5 - x^4 - 187x^3 - 359x^2 + 186x + 360 = 0.$

9. $x^6 - 10x^5 + 19x^4 + 110x^3 - 536x^2 + 800x - 384 = 0.$

10. If an equation involves only even powers of x, and the signs are all positive, the equation has no real root.

11. If an equation involves only odd powers of x, and the signs are all positive, the equation has the root 0, and no other real root.

?

12. Show that the equation

$$x^6 - 3x^2 - x + 1$$

has at least two imaginary roots.

13. Show that the equation

$$x^4 + 15x^2 + 7x - 11 = 0$$

has two imaginary roots, and determine the signs of the real roots.

14. Show that the equation $x^3 + qx + r = 0$ has one negative and two imaginary roots when q and r are both positive; and determine the character of the roots when q is negative and r positive.

15. Show that the equation $x^n - 1 = 0$ has but two real roots, $+1$ and -1, when n is even; and but one real root, $+1$, when n is odd.

16. Show that the equation $x^n + 1 = 0$ has no real root when n is even; and but one real root, -1, when n is odd.

---

482. Limits of the Roots. In solving numerical equations it is often desirable to obtain numbers between which the roots lie. Such numbers are called **limits of the roots.**

A *superior limit* of the positive roots of an equation is a number greater than any positive root. An *inferior limit* to the positive roots of an equation is a positive number less than any positive root.

General methods for finding limits to the roots are given in most text-books; but in practice close limits are more easily found as follows:

(1) $$x^4 - 5x^3 + 40x^2 - 8x + 23 = 0.$$

Writing this $\quad x^3(x-5) + 8x(x-5) + 23 = 0,$

we see that the left-member is positive for all values of x as great as 5; consequently, it cannot become 0 for any value as great as 5, and there is no root as great as 5.

5x−1

(2) $$x^5+3x^4+x^3-8x^2-51x+18=0.$$

Writing this $x^2(x^3-8)+3x(x^3-17)+x^3+18=0,$

we see that the left-member is positive for all values of x as great as 3; consequently there is no positive root as great as 3.

Sometimes we can find close limits by distributing the highest positive powers of x among the negative terms.

(3) $$x^4+x^3-2x^2-4x-24=0.$$

Multiplying by 2, $2x^4+2x^3-4x^2-8x-48=0.$

Writing this $x^2(x^2-4)+2x(x^2-4)+x^4-48=0,$

we see that there is no positive root as great as 3.

An inferior limit to the positive roots is found by putting $x=\frac{1}{y}$ (§ 469), and finding a superior limit to the positive roots of the transformed equation.

Limits to the *negative* roots of the equation $f(x)=0$ are found by finding limits to the *positive* roots of the equation $f(-x)=0$ (§ 466).

Exercise 79.

Find superior limits to the positive roots of the following equations:

1. $x^3-2x^2+4x+3=0.$
2. $2x^4-x^2-x+1=0.$
3. $3x^4+5x^3-12x^2+10x-18=0.$
4. $4x^4-3x^3-x^2+7x+5=0.$
5. $x^4-x^3-2x^2-4x-24=0.$
6. $4x^5-8x^4+22x^3+90x^2-60x+1=0.$

CHAPTER XXX.

NUMERICAL EQUATIONS.

483. A real root of a numerical equation is either **commensurable** or **incommensurable.**

Commensurable roots are either integers or fractions. Repeating decimals can be expressed as fractions (§ 231), and roots in that form are consequently commensurable.

Incommensurable roots cannot be found exactly, but may be calculated to any desired degree of accuracy by the method of approximation explained in this chapter.

COMMENSURABLE ROOTS.

484. Integral Roots. The process of finding integral roots given in § 441 is long and tedious when there are many numbers to be tried. The number of divisors to be tried is diminished by the following theorem:

Every integral root of an equation with integral coefficients is a divisor of the last term.

We shall prove this for an equation of the fourth degree, but the proof is perfectly general.

Let h be an integral root of the equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

where the coefficients a, b, c, d, e are all integers.

Since h is a root,

$$ah^4 + bh^3 + ch^2 + dh + e = 0, \qquad (\S\ 432)$$

or, $$e = -dh - ch^2 - bh^3 - ah^4.$$

Dividing by h,

$$\frac{e}{h} = -d - ch - bh^2 - ah^3.$$

Since the right-member is an integer, the left-member must be an integer. That is, e is divisible by h.

Similarly, for any equation with integral coefficients.

Hence, in applying the method of § 441, we need try only divisors of the last term. The necessary labor may be still further reduced by the method of the following section.

485. Newton's Method of Divisors. In the equation above $\frac{e}{h}$ is an integer. Put $\frac{e}{h} = D$, transpose d, and divide by h. Then,

$$\frac{D+d}{h} = -c - bh - ah^2.$$

Since the right-member is an integer, $D + d$ must be divisible by h.

Put $\frac{D+d}{h} = C$, transpose $-c$, and divide by h. Then,

$$\frac{C+c}{h} = -b - ah.$$

As before, $C + c$ must be divisible by h.

Put $\frac{C+c}{h} = B$, transpose $-b$, and divide by h. Then,

$$\frac{B+b}{h} = -a.$$

As before, $B + b$ must be divisible by h. Transposing $-a$, we have

$$\frac{B+b}{h} + a = 0,$$

provided h is a root.

The preceding gives the following rule:

Divide the last term by **h**; *if the quotient is an integer, to it add the preceding coefficient, and again divide by* **h**; *if this quotient is an integer, to it add the preceding coefficient; and so on.*

If h is a root, the quotients will all be integral, and the last sum will be zero. A failure in either respect implies that h is not a root.

From the above we also obtain

$$D = -(ah^3 + bh^2 + ch + d),$$
$$C = -(ah^2 + bh + c),$$
$$B = -(ah + b),$$

so that the successive quotients, with their signs changed, are (§ 436), in reversed order, the coefficients of the quotient obtained by dividing the left-member by $x - h$.

The above evidently applies to an equation of any degree.

Ex. Find the integral roots of

$$3x^4 - 23x^3 + 42x^2 + 32x - 96 = 0.$$

By substitution neither $+1$ nor -1 is a root.
The other divisors of -96 are ± 2, ± 3, ± 4, ± 6, etc.

Try $+2$:

$$\begin{array}{rrrrr|l} -96 & +32 & +42 & -23 & +3 & \underline{2} \\ & -48 & -8 & +17 & -3 & \\ \hline & -16 & +34 & -6 & 0 & \end{array}$$

Hence $+2$ is a root. The coefficients of the depressed equation in reversed order are $\quad -48 \quad -8 \quad +17 \quad -3$

Try $+2$ again:

$$\begin{array}{rrrr|l} -48 & -8 & +17 & -3 & \underline{2} \\ & -24 & -16 & & \\ \hline & -32 & +1 & & \end{array}$$

Since 2 is not a divisor of $+1$, $+2$ is not again a root.

Try -2:

$$\begin{array}{rrrr|l} -48 & -8 & +17 & -3 & \underline{-2} \\ & +24 & -8 & & \\ \hline & +16 & +9 & & \end{array}$$

and -2 is not a root.

Try $+3$:

$$\begin{array}{rrrr|l} -48 & -8 & +17 & -3 & \underline{3} \\ & -16 & -8 & +3 & \\ \hline & -24 & +9 & 0 & \end{array}$$

Hence $+3$ is a root. The depressed equation is

$$3x^2 - 8x - 16 = 0,$$

of which the roots are 4 and $-\frac{4}{3}$. Therefore the roots of the given equation are 2, 3, 4, $-\frac{4}{3}$.

The advantage of this method over that of § 441 is that if the number tried is not a root, this fact is detected as soon as we come to a fractional quotient; whereas, in § 441, we have to complete the division before we decide whether the number tried is a root or not.

486. Fractional Roots. *A rational fraction cannot be a root of an equation with integral coefficients in the p form.*

If possible let $\frac{h}{k}$, where h and k are integers, and $\frac{h}{k}$ is in its lowest terms, be a root. Then,

$$\frac{h^n}{k^n} + p_1 \frac{h^{n-1}}{k^{n-1}} + p_2 \frac{h^{n-2}}{k^{n-2}} + \cdots\cdots p_n = 0.$$

Multiplying by k^{n-1} and transposing,

$$\frac{h^n}{k} = -p_1 h^{n-1} - p_2 h^{n-2} k - \cdots\cdots - p_n k^{n-1}.$$

Now the right-member is an integer; the left-member is a fraction in its lowest terms, since h^n and k have no common divisor as h and k have no common divisor (§ 350, V.). But a fraction in its lowest terms cannot be equal to an integer. Hence $\frac{h}{k}$, or any other rational fraction, cannot be a root.

The real roots of an equation with integral coefficients in the p form are, therefore, integral or incommensurable.

In case an equation has fractional roots, we can find them as follows:

Transform the equation into an equation with integral coefficients by multiplying the roots by some number m (§ 468). Find the integral roots of the transformed equation, and divide each by m.

Ex. Solve the equation

$$36x^4 - 55x^2 - 35x - 6 = 0.$$

Write this

$$x^4 - \tfrac{55}{36}x^2 - \tfrac{35}{36}x - \tfrac{1}{6} = 0.$$

Multiplying the roots by 6, we obtain

$$x^4 - 55x^2 - 210x - 216 = 0,$$

of which the roots are found to be $-2,\ -3,\ -4,\ 9.$

Hence, the roots of the given equation are

$$-\tfrac{2}{6},\ -\tfrac{3}{6},\ -\tfrac{4}{6},\ \tfrac{9}{6};\ \text{or,}\ -\tfrac{1}{3},\ -\tfrac{1}{2},\ -\tfrac{2}{3},\ \tfrac{3}{2}.$$

Exercise 80.

Find the commensurable roots, and if possible all the roots, of each of the following equations:

1. $x^4 - 4x^3 - 8x + 32 = 0.$
2. $x^3 - 6x^2 + 10x - 8 = 0.$
3. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$
4. $x^3 + 3x^2 - 30x + 36 = 0.$
5. $x^4 - 12x^3 + 32x^2 + 27x - 18 = 0.$
6. $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0.$
7. $x^5 - 5x^4 + 3x^3 + 17x^2 - 28x + 12 = 0.$
8. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

9. $x^5 - 8x^4 + 11x^3 + 29x^2 - 36x - 45 = 0.$

10. $x^5 - x^4 - 6x^3 + 9x^2 + x - 4 = 0.$

11. $2x^4 - 3x^3 - 20x^2 + 27x \cancel{-} 18 = 0.$ $+$

12. $2x^4 - 9x^3 - 27x^2 + 134x - 120 = 0.$

13. $x^6 + 3x^5 - 2x^4 - 15x^3 - 15x^2 + 8x + 20 = 0.$

14. $18x^3 + 3x^2 - 7x - 2 = 0.$

15. $24x^3 - 34x^2 - 5x + 3 = 0.$

16. $27x^3 - 18x^2 - 3x + 2 = 0.$

17. $18x^4 + 9x^3 + 10x^2 - 8x + 1 = 0.$

18. $36x^4 + 48x^3 - 23x^2 - 17x + 6 = 0.$

INCOMMENSURABLE ROOTS.

487. Location of the Roots. In order to calculate the value of an incommensurable root we must first find a rough approximation to the value of the root; for example, two integers between which it lies. This can generally be accomplished by successive applications of the principle of § 477. In some equations the methods of § 479–482 may be useful.

(1) Consider the equation

$$x^3 - 6x^2 + 3x + 5 = 0.$$

We find (§ 436), $f(0) = +\ 5;\quad f(4) = -15;$
$f(1) = +\ 3;\quad f(5) = -\ 5;$
$f(2) = -\ 5;\quad f(6) = +23;$
$f(3) = -13;\quad f(-1) = -5.$

All numbers above 6 give $+$; all below -1, give $-$.

From the above (§ 477) the three roots are all real; one between 1 and 2; one between 5 and 6; one between 0 and -1.

(2) The equation

$$x^4 - 2x^3 - 11x^2 + 6x + 2 = 0,$$

has, by Descartes's rule (§ 479), not more than two positive roots and not more than two negative roots.

We find (§ 436),
$$\begin{aligned} f(0) &= +\ 2; & f(5) &= +132; \\ f(1) &= -\ 4; & f(-1) &= -\ 12; \\ f(2) &= -30; & f(-2) &= -\ 22; \\ f(3) &= -52; & f(-3) &= +\ 20; \\ f(4) &= -22; & f(-4) &= +186. \end{aligned}$$

Hence there are two positive roots, one between 0 and 1, and one between 4 and 5; and two negative roots, one between 0 and -1, and one between -2 and -3.

Let us find more closely a value for the root between 0 and 1. We find $f(0.5) = +2.06+$. Since $f(1) = -4$, the root lies between 0.5 and 1.

Try 0.8: we find $f(0.8) = -0.9+$. Hence the root lies between 0.5 and 0.8.

We find $f(0.7) = +0.4+$. Hence the root lies between 0.7 and 0.8.

In a similar manner we find the root between 0 and -1 to lie between -0.2 and -0.3.

The first significant figures of the roots are accordingly 0.7, 4, -0.2, -2.

Exercise 81.

Determine the first significant figure of each real root of the following equations:

1. $x^3 - x^2 - 2x + 1 = 0.$
2. $x^3 - 5x - 3 = 0.$
3. $x^3 - 5x^2 + 7 = 0.$
4. $x^3 + 2x^2 - 30x + 39 = 0.$
5. $x^3 - 6x^2 + 3x + 5 = 0.$
6. $x^3 + 9x^2 + 24x + 17 = 0.$
7. $x^3 - 15x^2 + 63x - 50 = 0.$
8. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$

488. Horner's Method, Positive Roots. Suppose the first figure of the root to have been found. Any number of remaining figures may be calculated by the method of approximation known as Horner's Method.

We proceed to illustrate the process by an example.

Take the equation

$$x^3 - 6x^2 + 3x + 5 = 0. \qquad (1)$$

By § 487, Ex. 1, one root of this equation lies between 1 and 2. We proceed to calculate that root.

Diminish the roots by 1 (§ 471):

$$\begin{array}{rrrrl} 1 & -6 & +3 & +5 & \underline{|1} \\ & \underline{+1} & \underline{-5} & \underline{-2} & \\ & -5 & -2 & +\mathbf{3} & \\ & \underline{+1} & \underline{-4} & & \\ & -4 & -\mathbf{6} & & \\ & \underline{+1} & & & \\ & -\mathbf{3} & & & \end{array}$$

The transformed equation is, therefore,

$$y^3 - 3y^2 - 6y + 3 = 0. \qquad (2)$$

The roots of equation (2) are each less by 1 than the roots of equation (1). Equation (1) has a root between 1 and 2; equation (2) has, therefore, a root between 0 and 1. Since this root is less than 1, y^3 and y^2 are both less than y. Neglecting these terms, we have

$$-6y + 3 = 0, \text{ or } y = 0.5.$$

At this stage of the process the figure thus obtained will not in general be the correct one. If, however, we neglect only the y^3 term, we obtain

$$-3y^2 - 6y + 3 = 0,$$

$$y^2 + 2y - 1 = 0,$$

of which one root is $\sqrt{2} - 1 = 0.4+$.

We can also find the second figure of the root as follows:

Take the first value 0.5.

With this assumed value of y, computing the value of $y^3 - 3y^2$, and substituting, we obtain $6y = 2.375$; whence $y = 0.4$, approximately.

We now diminish the roots of (2) by 0.4:

$$\begin{array}{rrrr|l}
1 & -3 & -6 & +3 & 0.4 \\
 & +0.4 & -1.04 & -2.816 & \\
\hline
 & -2.6 & -7.04 & +\mathbf{0.184} & \\
 & +0.4 & -0.88 & & \\
\hline
 & -2.2 & -\mathbf{7.92} & & \\
 & +0.4 & & & \\
\hline
 & -\mathbf{1.8} & & &
\end{array}$$

The second transformed equation is

$$z^3 - 1.8z^2 - 7.92z + 0.184 = 0. \qquad (3)$$

The roots of (3) are less by 0.4 than those of (2), and less by 1.4 than those of (1). Equation (2) has a root between 0.4 and 0.5; equation (3) has, therefore, a root between 0 and 0.1.

Since this root is much less than 1, we shall probably obtain a correct value for the next figure of the root by neglecting the z^3 and z^2 terms in equation (3).

This gives $-7.92z + 0.184 = 0$; whence $z = 0.02+$.

Diminish the roots of (3) by 0.02:

$$\begin{array}{rrrr|l}
1 & -1.8 & -7.92 & +0.184 & 0.02 \\
 & +0.02 & -0.0356 & -0.159112 & \\
\hline
 & -1.78 & -7.9556 & +\mathbf{0.024888} & \\
 & +0.02 & -0.0352 & & \\
\hline
 & -1.76 & -\mathbf{7.9908} & & \\
 & +0.02 & & & \\
\hline
 & -\mathbf{1.74} & & &
\end{array}$$

The third transformed equation is

$$u^3 - 1.74u^2 - 7.9908u + 0.024888 = 0. \qquad (4)$$

The roots of (4) are less by 0.02 than those of (3), and less by 1.42 than those of (1).

Neglecting the u^3 and u^2 terms, we obtain $u = 0.0031+$,

so that to four places of decimals the root of (1) is 1.4231. The process may evidently be continued until the root is calculated to any desired degree of accuracy.

489. We shall now make some observations on the preceding work.

First: If we diminish the roots by a number *less* than the required root, as we do not pass through the root, the sign of the last term remains unchanged throughout the work. The last coefficient but one will always have a sign opposite to that of the last term.

If, in (3), the signs of the last two terms were alike, the value of z would be $-0.02+$. This would show that the value assumed for z was too great, and we should diminish the value of z and make the last transformation again. In beginning an example, one is very likely to assume too large a value for the next figure of the root; in solving (2), for instance, the first solution gave $y = 0.5$, and had that value been tried, it would have proved to be too great.

REMARK. The *first* transformation may, however, change the sign of the last term. Thus, if there had been a root between 0 and 1 in equation (1), diminishing the roots by 1 would have changed the sign of the last term.

Second: In finding the second figure of the root we make use of the last three terms of the first transformed equation instead of the last two terms. Or, we may use the alternative method. One of these methods will generally give the correct figure. In any case we can find the correct figure by another trial.

Any figure after the second is generally found correctly from the last two terms; for, in this case, the root is small and its square and cube so much smaller than the root itself that the terms in which they appear have but slight influence upon the result.

490. It is not necessary to write out the successive transformed equations. When the coefficients of any transformed

equation have been computed, the next figure of the root may be found by dividing the last coefficient by the preceding coefficient, and changing the sign of the quotient.

Thus, in equation (4), the next figure of the root is obtained by dividing 0.024888 by 7.9908.

On this account the last coefficient but one of each transformed equation is called a **trial divisor.**

Sometimes the last coefficient but one in one of the transformed equations is zero. To find the next figure of the root in this case follow the method given for finding the second figure of the root.

The work may now be collected and arranged as follows:

1	−6	+3	+5	1.423+
	+1	−5	−2	
	−5	−2	**+3**	
	+1	−4	−2.816	
	−4	**−6**	**+0.184**	
	+1	−1.04	−0.159112	
	−3	−7.04	**+0.024888**	
	+0.4	−0.88		
	−2.6	**−7.92**		
	+0.4	−0.0356		
	−2.2	−7.9556		
	+0.4	−0.0352		
	−1.8	**−7.9908**		
	+0.02			
	−1.78			
	+0.02			
	−1.76			
	+0.02			
	−1.74			

The broken lines mark the conclusion of each transformation. The numbers in heavy type are the coefficients of the successive transformed equations, the first coefficient of each equation being the same as the first coefficient of the given equation. In this example the first coefficient is 1.

When we have obtained the root to three places of decimals we can generally obtain two to three more figures of the root by simple division.

491. In practice it is convenient to avoid the use of the decimal points. We can do this as follows: multiply the roots of the first transformed equation by 10, the roots of the second transformed equation by 100, and so on. In the last example the first transformed equation will now be

$$y^3 - 30y^2 - 600y + 3000 = 0,$$

and this equation will have a root between 4 and 5. The second transformed equation will now be

$$z^3 - 180z^2 - 79{,}200z + 184{,}000 = 0,$$

and this equation will have a root between 2 and 3. And so on.

Comparing these equations with the equations in § 488, we see that we can avoid the use of the decimal point by adopting the following rule:

When the coefficients of a transformed equation have been obtained, add one cipher to the second coefficient, two ciphers to the third coefficient, and so on. The coefficients and the next figure of the root are now integers. The work proceeds as in § 490.

If the root of the given equation lay between 0 and 1, we should *begin* by multiplying the roots of the given equation by 10.

The complete work of the last example, for six figures of the root, will now be as follows:

1	−6	+3	+5	1.42311+
	+1	−5	−2	
	−5	−2	**+3000**	
	+1	−4	−2816	
	−4	**−600**	**+184000**	
	+1	−104	−159112	
	−30	−704	**+24888000**	
	+4	−88	−23988033	
	−26	**−79200**	**+899967000**	
	+4	−356	−800138609	
	−22	−79556	**+99828391**	
	+4	−352		
	−180	**−7990800**		
	+2	−5211		
	−178	−7996011		
	+2	−5202		
	−176	**−800121300**		
	+2	−17309		
	−1740	−800138609		
	+3	−17308		
	−1737	**−800155917**		
	+3			
	−1734			
	+3			
	−17310			
	+1			
	−17309			
	+1			
	−17308			
	+1			
	−17307			

We can find five more figures of the root by simple division. If we divide 800,155,917 by 99,828,391, we obtain 0.124761, so that the required root to ten places of decimals is 1.4231124761.

The reason is seen by examining the last transformed equation. Write this

$$8.00155917\,w = 0.000099828391 - 1.7307\,w^2 + w^3.$$

As w is about 0.00001, w^2 is about 0.0000000001, and w^3 is still smaller. Hence the error in neglecting the w^2 and w^3 terms is in $8\,w$ about 0.00000000017, and in w about 0.00000000002. The result obtained by division will therefore be true to ten places of decimals.

492. Negative Roots. To avoid the inconvenience of working with negative numbers, when we wish to calculate a negative root, we change the signs of the roots (§ 466), and calculate the corresponding positive roots of the transformed equation.

Thus one root of the equation

$$x^3 - 6x^2 + 3x + 5 = 0$$

lies between 0 and -1 (§ 487). By Horner's Method we find the corresponding root of

$$x^3 + 6x^2 + 3x - 5 = 0$$

to be 0.6696+. Hence, the required root of the given equation is $-0.6696+$.

Exercise 82.

Compute for each of the following equations the root of which the first figure is the number in parenthesis opposite the equation. Carry out the work to three places of decimals :

1. $x^3 + 3x - 5 = 0$ (1).
2. $x^3 - 6x - 12 = 0$ (3).
3. $x^3 + x^2 + x - 100 = 0$ (4).
4. $x^3 + 10x^2 + 6x - 120 = 0$ (2).
5. $x^3 + 9x^2 + 24x + 17 = 0$ (−4).
6. $x^4 - 12x^3 + 12x - 3 = 0$ (−3).
7. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$ (−0.).

493. Contraction of Horner's Method. In § 491 the student will see that if we seek only the first six figures of the root, the last six figures of the fourth coefficient of the last transformed equation may be rejected without affecting the result. Those figures of the second and third coefficients which enter into the fourth coefficient only in the rejected figures may also be rejected. Moreover, we may reject all the figures which stand in vertical lines over the figures already rejected.

The work may now be conducted as follows:

1	-6	$+3$	$+5$	1.42311+
	$+1$	-5	-2	
	-5	-2	$+3000$	
	$+1$	-4	-2816	
	-4	-600	$+184000$	
	$+1$	-104	-159112	
	-30	-704	$+24888$	
	$+4$	-88	-23991	
	-26	-79200	$+897$	
	$+4$	-356	-800	
	-22	-79556	$+97$	
	$+4$	-352	-80	
	-180	-79908		
	$+2$	-7991		
	-178	-6		
	$+2$	-7997		
	-176	-6		
	$+2$	-8003		
	-174	-800		
	-2	-80		

The double lines in the first column indicate that beyond this stage of the work the first column disappears altogether.

In the present example we find three figures of the root before we begin to contract. We then contract the work as follows:

Instead of adding ciphers to the coefficients of the transformed equation, we leave the last term as it is; from the last coefficient but one we strike off the last figure; from the last coefficient but two we strike off the last two figures; and so on. In each case we take for the remainder the nearest integer. Thus, in the first column of the preceding example we strike off from 174 the last two figures, and take for the remainder 2 instead of 1.

The contracted process soon reduces to simple division. Thus, in the last example, the last two figures of the root were found by simply dividing 897 by 800.

To insure accuracy in the last figure, the last divisor must consist of at least two figures. Consider the trial divisor at any stage of the work. If we begin to contract, we strike off one figure from the trial divisor *before* finding the next figure of the root. Since the last divisor is to consist of two figures, the contracted process will give us two less figures than there are figures in the trial divisor.

Thus, in § 491, if we begin to contract at the third trial divisor, $-79{,}908$, we can obtain three more figures of the root; if we begin to contract at the fourth trial divisor, $-800{,}213$, we can obtain four more figures of the root; and so on.

The student should carefully compare the contracted process on page 450 with the uncontracted on page 448.

494. When the root sought is a large number, we cannot find the successive figures of its *integral* portion by dividing the absolute term by the preceding coefficient, because

the neglect of the higher powers, which are in this case large numbers, leads to serious error.

Let it be required to find one root of

$$x^4 - 3x^2 + 11x - 4{,}842{,}624{,}131 = 0. \qquad (1)$$

By trial, we find that a root lies between 200 and 300. Diminishing the roots of (1) by 200, we have

$$y^4 + 800y^3 + 239{,}997y^2 + 31{,}998{,}811y - 3{,}242{,}741{,}931 = 0. \qquad (2)$$

$$\text{If } y = 60, \quad f(y) = -273{,}064{,}071.$$

$$\text{If } y = 70, \quad f(y) = +471{,}570{,}139.$$

The signs of $f(y)$ show that a root lies between 60 and 70. Diminishing the roots of (2) by 60, we obtain

$$z^4 + 1040z^3 + 405{,}597z^2 + 70{,}302{,}451z - 273{,}064{,}071 = 0. \qquad (3)$$

The root of this equation is found by trial to lie between 3 and 4. Diminishing the roots by 3, we may find the remaining figures of the root by the usual process.

495. Any root of a number can be extracted by Horner's Method

Ex. Find the fourth root of 473.

Here $\qquad x^4 = 473,$

or $\qquad x^4 + 0x^3 + 0x^2 + 0x - 473 = 0.$

Calculating the root, $\quad x = 4.66353+.$

If the number be a perfect power, the root will be obtained exactly.

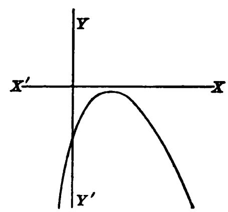
496.* Roots nearly Equal. In the preceding examples the changes of sign in the value of $f(x)$ enable us to determine the situation of the roots. In rare cases two roots may be so nearly equal that they both lie between consecutive integers. In this case the existence of the roots will not be indicated by a change of sign in $f(x)$, and we must resort to other means to detect their presence.

Ex. Consider the equation

$$x^3 - 515x^2 + 1155x - 649 = 0. \tag{1}$$

By Descartes' rule this equation has no negative root. It has therefore certainly one, and perhaps three, positive roots.

We find
$$\begin{aligned} f(-1) &= -2320; \\ f(0) &= -\ \ 649; \\ f(1) &= -\ \ \ \ 8; \\ f(2) &= -\ \ 391; \\ f(3) &= -1792. \end{aligned}$$



The approach of $f(x)$ towards 0 indicates either that there are two roots near 1, or that the function approaches 0 without reaching it; the graph in the latter case being as here given.

Let us proceed on the supposition that two roots near 1 do exist. Diminish the roots by 1. The transformed equation

$$y^3 - 512y^2 + 128y - 8 = 0, \tag{2}$$

by Descartes' rule, still has either one or three positive roots, so that we have not passed the roots.

If we had diminished the roots by 2, we should have obtained

$$y^3 - 509y^2 - 893y - 391 = 0,$$

which has but one positive root; so that we have passed both roots.

To find the second figure of the root, neglect the first term of equation (2). Since the roots are nearly equal, the expression

$$512y^2 - 128y + 8$$

must be nearly a perfect square. Comparing this with $a(y-\alpha)^2$, or $ay^2 - 2a\alpha y + a\alpha^2$, we see that $\dfrac{128}{2 \times 512}$ and $\dfrac{2 \times 8}{128}$ are approximate values for the roots; these both give $\frac{1}{8}$, or 0.12.

Diminish the roots by 0.1; the work is as before. Continue until the two quotients obtained as above give different numbers for the next figure of the root. In the present example this occurs when we come to the third decimal figure; the transformed equation is

$$u^3 - 51{,}164u^2 + 51{,}632u - 11{,}072 = 0,$$

and the two quotients are 0.5 + and 0.3 +. To separate the roots, try 0.4; the left-member of the last equation is found to be +. Since 0 gives − and 1 gives −, there is one root between 0 and 0.4, and one between 0.4 and 1.

To calculate the first root, we try 0.3; as this gives a − sign we diminish the roots by 0.3, and proceed as in § 493.

1	−515	+1155	−649	1.1230907
	1	−514	+641	
	−514	+641	**−8000**	
	1	−513	+7681	
	−513	**+12800**	**−319000**	
	1	−5119	+307928	
	−5120	+7681	**−11072000**	
	1	−5118	+10885167	
	−5119	**+256300**	**−186833**	
	1	−102336	+184284	
	−5118	+153964	−1549	
	1	−102332	+1400	
	−51170	**+5163200**	−149	
	2	−1534911		
	−51168	+3628389		
	2	−1534902		
	−51166	**+2093487**		
	2	+209349		
	−511640	+20935		
	3	−459		
	−511637	+20476		
	3	−459		
	−511634	+20017		
	3	2002		
	−511631	200		
	−5116			
	−51			

To calculate the second root, we return to the equation

$$u^3 - 51{,}164\,u^2 + 51{,}632\,u - 11{,}072 = 0.$$

We have $f(0.4) = +$, $f(1) = -$; we find $f(0.6) = +$, $f(0.7) = +0.383$. Since $f(0.7)$ is so small, $f(0.8)$ is undoubtedly negative.

Diminish the roots by 7 and proceed as follows:

$$\begin{array}{llrrrl}
1 & & \mathbf{-511640} & \mathbf{+5163200} & \mathbf{-11072000} & \underline{|1.1270002} \\
 & & 7 & -3581431 & +11072383 & \\
\hline
 & & -511633 & +1581769 & \mathbf{+383} & \\
 & & 7 & -3581382 & & \\
\hline
 & & -511626 & \mathbf{-1999613} & & \\
 & & 7 & -200 & & \\
\hline
 & & \mathbf{-511619} & & & \\
\end{array}$$

Since the sum of the roots (§ 442) is 515, we can find the third root by subtracting from 515 the sum of the two roots already found.

$$\begin{array}{ll}
\text{1st root,} & 1.1230907 \\
\text{2d root,} & \underline{1.1270002}
\end{array}$$

$$515 - 2.2500909 = 512.7499091, \text{ 3d root.}$$

497. From the preceding sections we obtain the following general directions for solving a numerical equation:

I. Find and remove commensurable roots by §§ 484–486, if there are any such roots in the equation.

II. Determine the situation and thence the first figure of each of the incommensurable roots as in § 487.

III. Calculate the incommensurable roots by Horner's Method.

Exercise 83.

Calculate to six places of decimals the positive roots of the following equations:

1. $x^3 - 3x - 1 = 0.$
2. $x^3 + 2x^2 - 4x - 43 = 0.$

3. $3x^3+3x^2+8x-32=0.$
4. $2x^3-26x^2+131x-202=0.$
5. $x^4-12x+7=0.$
6. $x^4-5x^3+2x^2-13x+55=0.$

Calculate, to six places of decimals where incommensurable, the real roots of the following equations:

7. $x^3=35{,}499.$
8. $x^3=242{,}970{,}624.$
9. $x^4=707{,}281.$
10. $x^5=147{,}008{,}443.$
11. $x^3+2x+20=0.$
12. $x^3-10x^2+8x+120=0.$

Each of the following equations has two roots nearly equal. Calculate them to six places of decimals:

13.* $x^3-x^2-6x+13=0.$
14.* $2x^4+8x^3-35x^2-36x+117=0.$
15.* $x^3+11x^2-102x+181=0.$

STURM'S THEOREM.

498. The problem of determining the number and situation of the real roots of an equation is completely solved by Sturm's Theorem. In theory Sturm's method is perfect; in practice its application is long and tedious. For this reason, the situation of the roots is in general more easily determined by the methods already given.

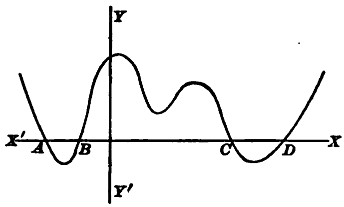
Before passing on to Sturm's Theorem itself we shall prove two preliminary theorems.

499. Situation of the Roots $\mathbf{f'(x)=0}$. *Between any two distinct real roots of the equation* $f(x)=0$ *there lies at least one real root of the equation* $f'(x)=0$.

Let α and β be two real roots of $f(x)=0$, β being greater than α. Then $f(\alpha)=0$ and $f(\beta)=0$. As x in-

creases continuously from α to β, $f(x)$ changes from 0 to 0 again; and must first increase and then decrease, or first decrease and then increase. Hence, there must be some point at which $f'(x)$ changes from + to −, or *vice versa*. Therefore, for some value of x between α and β, $f'(x)$ must be zero. Hence, at least one root of $f'(x) = 0$ must lie between α and β.

In the graph the curve will be horizontal where $f'(x) = 0$. In the figure here given, A, B, C, D correspond to roots of $f(x) = 0$. Between A and B there is one root of $f'(x) = 0$; between B and C, three roots; and between C and D, one root.



It is evident that if more than one root of $f'(x)$ lies between α and β, the number of roots must be an *odd* number.

500. Signs of f(x) and f'(x). Let α *be any real root of an equation*, $f(x) = 0$, *which has no equal roots.*

Let x *change continuously from* $\alpha - h$, *a value a little less than* α, *to* $\alpha + h$, *a value a little greater than* α. *Then* $f(x)$ *and* $f'(x)$ *will have unlike signs immediately before* x *passes through the root, and unlike signs immediately after* x *passes through the root.*

For $$f(\alpha - h) = -hf'(\alpha) + \frac{h^2}{\lfloor 2}f''(\alpha) - \cdots\cdots,$$

and $$f'(\alpha - h) = f'(\alpha) - hf''(\alpha) + \cdots\cdots; \quad (\S\ 463)$$

since $f(\alpha) = 0$, as α is a root of $f(x) = 0$.

When h is very small, the sign of each series on the right will be the sign of its first term (§ 475); and $f(\alpha - h)$ and $f'(\alpha - h)$ will evidently have opposite signs.

Similarly, $f(\alpha + h)$ and $f'(\alpha + h)$ will have like signs.

The above is also evident from the graph of $f(x)$.

501. Sturm's Functions. The process of finding the H. C. F. of $f(x)$ and $f'(x)$ has been employed (§ 462) in obtaining the multiple roots of the equation $f(x)=0$. We use the same process in Sturm's Method.

Let $f(x)=0$ be an equation which has no multiple roots; let the operation of finding the H. C. F. of $f(x)$ and $f'(x)$ be carried on until the remainder does not involve x, *the sign of each remainder obtained being changed before it is used as a divisor.*

If there is a H. C. F., the equation has multiple roots. Remove them and proceed with the reduced equation.

Represent by $f_2(x)$, $f_3(x)$, ... $f_n(x)$ the several remainders with their signs changed. These expressions with $f'(x)$ are called **Sturm's Functions.**

Now, if D represent the dividend, d the divisor, q the quotient, and R the remainder,

$$D \equiv qd + R.$$

Consequently,
$$\begin{aligned}
f(x) &\equiv q_1 f'(x) - f_2(x),\\
f'(x) &\equiv q_2 f_2(x) - f_3(x),\\
f_2(x) &\equiv q_3 f_3(x) - f_4(x),\\
&\cdots\cdots\cdots\\
&\cdots\cdots\cdots\\
f_{n-2}(x) &\equiv q_{n-1} f_{n-1}(x) - f_n(x);
\end{aligned}$$

where q_1, q_2, ... q_{n-1} represent the several quotients, or the quotients multiplied by positive integers.

From the above identities we have the following:

I. Two consecutive functions cannot vanish for the same value of x.

For example, suppose $f_2(x)$ and $f_3(x)$ to vanish for a particular value of x. Give to x this value in all the identities. By the third identity, $f_4(x)$ will vanish; by the

fourth, $f_5(x)$ will vanish; finally, $f_n(x)$ will vanish, which is contrary to the hypothesis that $f(x)=0$ has no multiple roots.

II. When we give to x a value which causes any one function to vanish, the adjacent functions have opposite signs.

Thus, if $f_3(x)=0$, from the third identity $f_2(x)=-f_4(x)$.

502. Sturm's Theorem. We are now in a position to enunciate Sturm's Theorem:

If in the series of functions

$$f(x),\ f'(x),\ f_2(x)\cdots\cdots f_n(x)$$

we give to x *any particular value* a, *and determine the number of variations of sign; then give to* x *any greater value* b, *and determine the number of variations of sign; the number of variations lost is the number of real roots of the equation* $f(x)=0$ *between* a *and* b.

For, let x increase continuously from a to b.

First: Take the case in which x passes through a root of any of the functions $f'(x), f_2(x)\cdots\cdots f_{n-1}(x)$, for example $f_4(x)$. The adjacent functions in this case have opposite signs. $f_4(x)$ itself changes sign, but this has no effect on the number of variations; for if just before x passes through the root the signs are $+\,+\,-$, just after x passes through the root they will be $+\,-\,-$, and the number of variations is in each case one.

Hence, there is no change in the number of variations of sign when x passes through a root of any of the functions $f'(x),\ f_2(x),\cdots\cdots f_{n-1}(x)$.

Second: Take the case in which x passes through a root of $f(x)=0$. Since $f(x)$ and $f'(x)$ have unlike signs just before x passes through the root, and like signs just after (§ 500), there is one variation lost for each root of $f(x)=0$.

Hence, the number of real roots between a and b is the number of variations of sign lost as x passes from a to b.

To determine the total number of real roots, we take x first very large and negative, and then very large and positive. The sign of each function is then the sign of its first term (§ 474).

The student may not understand how it is that $f(x)$ and $f'(x)$ always have unlike signs just before x passes through a root.

Let α and β be two consecutive roots of $f(x)=0$; let h be very small. Suppose that at α $f(x)$ changes from $+$ to $-$; then $f'(\alpha)$ is $-$ (§ 460).

When $\quad x=\alpha-h,\quad f(x)=+,\quad f'(x)$ is $-$;
$\qquad\quad x=\alpha,\quad f(x)=0,\quad f'(x)$ is $-$.

As x changes from α to β, $f'(x)$ passes through an odd number of roots (§ 499), and consequently changes sign. Hence, when $x=\beta-h$, $f(x)$ is $-$, $f'(x)$ is $+$; and $f'(x)$ and $f(x)$ again have unlike signs.

503. Examples. (1) Determine the number and signs of the real roots of the equation

$$x^4-4x^3+6x^2-12x+1=0.$$

Here $\qquad f'(x)\equiv 4x^3-12x^2+12x-12.$

Let us take for $f'(x)$, however, the simpler expression

$$x^3-3x^2+3x-3.$$

We proceed as if to find the H. C. F., changing the sign of each remainder before using it as a divisor.

$1-3+3-3$	$1-4+6-12+1$	$1-1$
$3-9+9-9$	$1-3+3-3$	
$3+1$	$-1+3-9+1$	
$-10+9$	$-1+3-3+3$	
$-30+27$	$-6-2$	
$-30-10$	$3+1$	$1-10+37$
$37-9$		
$111-27$		
$111+37$		
-64		
$+64$		

The coefficients of the several functions are in heavy type. In the ordinary process of finding the highest common factor we can change signs at pleasure. In finding Sturm's functions we cannot do this as the sign is all important. We can, however, take out any positive factor.

We now have
$$\begin{aligned} f(x) &\equiv x^4 - 4x^3 + 6x^2 - 12x + 1,\\ f'(x) &\equiv x^3 - 3x^2 + 3x - 3,\\ f_2(x) &\equiv 3x + 1,\\ f_3(x) &\equiv +64. \end{aligned}$$

When	$f(x)$	$f'(x)$	$f_2(x)$	$f_3(x)$	
$x = -1000$	$+$	$-$	$-$	$+$	2 variations.
$x = 0$	$+$	$-$	$+$	$+$	2 variations.
$x = +1000$	$+$	$+$	$+$	$+$	0 variations.

Hence, the equation has two real positive roots; it must therefore have two imaginary roots.

The real roots will be found by § 487 to lie one between 0 and 1, and one between 3 and 4.

(2) Investigate the reality of the roots of the equation

$$x^3 + 3Hx + G = 0.$$

We find
$$\begin{aligned} f(x) &\equiv x^3 + 3Hx + G,\\ f'(x) &\equiv 3(x^2 + H),\\ f_2(x) &\equiv -2Hx - G,\\ f_3(x) &\equiv -(G^2 + 4H^3). \end{aligned}$$

If $G^2 + 4H^3$ is *positive*, we have

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	$-$	$+$	$\pm$	$-$	2 variations.
$x = +\infty$	$+$	$+$	$\mp$	$-$	1 variation.

Since H may be either + or −, the sign of $f_2(x)$ is ambiguous. Hence, when $G^2 + 4H^3$ is positive, there is but one real root.

If $G^2 + 4H^3$ is *negative*, H must be negative, and we have

$x = -\infty$	$-$	$+$	$-$	$+$	3 variations.
$x = +\infty$	$+$	$+$	$+$	$+$	0 variation.

Hence, when $G^2 + 4H^3$ is negative, there are three real roots.

Exercise 84.

Determine by Sturm's Theorem the number and situation of the real roots of the following equations:

1. $x^3 - 4x^2 - 11x + 43 = 0.$
2. $x^3 - 6x^2 + 7x - 3 = 0.$
3. $x^4 - 4x^3 + x^2 + 6x + 2 = 0.$
4. $x^4 - 5x^3 + 10x^2 - 6x - 21 = 0.$
5. $x^4 - x^3 - x^2 + 6 = 0.$
6. $x^4 - 2x^3 - 3x^2 + 10x - 4 = 0.$
7. $x^5 + 2x^4 + 3x^3 + 3x^2 - 1 = 0.$
8. $x^5 + x^3 - 2x^2 + 3x - 2 = 0.$

CHAPTER XXXI.

GENERAL SOLUTION OF EQUATIONS.

504. Numerical and Algebraic Solutions. By the methods of the preceding chapter we can find to any desired degree of accuracy the real roots of a numerical equation of any degree. The methods are theoretically complete, and the solution of a numerical equation becomes simply a question of the labor required for the necessary computations.

In the case of a literal equation we have an entirely different problem to solve. To solve a literal equation, we have to find in terms of the coefficients expressions which will, when substituted for the unknown in the given equation, reduce that equation to an identity. Thus, the roots of the general quadratic have been found; they are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the case of a particular quadratic with numerical coefficients the roots can be found by putting for a, b, c in the above expression their particular values, and performing the indicated operations.

Similar solutions have been obtained for the general equations of the third and fourth degrees, and for certain special forms of equations of higher degrees.

The solution of the general equation of the fifth degree involves expressions called *elliptic functions*, and is consequently beyond the scope of the present treatise.

In many cases, however, the numerical values of the roots of a particular equation are not easily obtained from the general solution, and for numerical equations the general solutions are in such cases of little value.

A general solution differs from the solutions obtained in the last chapter in that a general solution represents not one particular root but all the roots indiscriminately.

We shall first consider equations of two special forms, reciprocal and binomial equations.

505. Reciprocal Equations. Reciprocal equations (§ 470), called also *recurring equations*, are of four forms:

(1) Degree even; corresponding coefficients equal with like signs.

(2) Degree even; corresponding coefficients numerically equal but with unlike signs.

(3) Degree odd; corresponding coefficients equal with like signs.

(4) Degree odd; corresponding coefficients numerically equal but with unlike signs.

The following are examples of the four forms:

(1) $2x^4 - 3x^3 + 4x^2 - 3x + 2 = 0;$

(2) $3x^6 - x^5 + 2x^4 - 2x^2 + x - 3 = 0;$

(3) $x^5 + 3x^4 - 2x^3 - 2x^2 + 3x + 1 = 0;$

(4) $2x^5 + 5x^4 + x^3 - x^2 - 5x - 2 = 0.$

Every equation of the second form will evidently want the middle term.

Every reciprocal equation of the second, third, or fourth form can be depressed to an equation of the first form.

Second Form: Consider the equation

$$ax^6 + bx^5 + cx^4 - cx^2 - bx - a = 0.$$

Writing this

$$a(x^6 - 1) + bx(x^4 - 1) + cx^2(x^2 - 1) = 0,$$

we see that the equation is divisible by $x^2 - 1$; consequently 1 and -1 are both roots. The depressed equation is evidently of the first form.

Similarly for any equation of the second form.

Third Form: Consider the equation

$$ax^5 + bx^4 + cx^3 + cx^2 + bx + a = 0.$$

Writing this

$$a(x^5 + 1) + bx(x^3 + 1) + cx^2(x + 1) = 0,$$

we see that the equation is divisible by $x + 1$; consequently -1 is a root. The depressed equation is evidently of the first form.

Similarly for any equation of the third form.

Fourth Form: Consider the equation

$$ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0.$$

Writing this

$$a(x^5 - 1) + bx(x^3 - 1) + cx^2(x - 1) = 0,$$

we see that the equation is divisible by $x - 1$; consequently $+1$ is a root. The depressed equation is evidently of the first form.

Similarly for any equation of the fourth form.

By the preceding, to solve any reciprocal equation, it is only necessary to solve one of the first form.

506. Any reciprocal equation of the first form can be depressed to an equation of half the degree. We proceed to illustrate by examples:

(1) Solve the equation

$$x^4 - 12x^3 + 29x^2 - 12x + 1 = 0.$$

Divide by x^2, $\quad x^2 + \dfrac{1}{x^2} - 12\left(x + \dfrac{1}{x}\right) + 29 = 0.$

Put $\quad z = x + \dfrac{1}{x};$

then $\quad z^2 - 2 - 12z + 29 = 0,$

or $\quad z^2 - 12z + 36 = 9,$

whence $\quad z = 9$ or 3.

Solving the equations

$$x + \frac{1}{x} = 9, \qquad x + \frac{1}{x} = 3,$$

we find $\quad x = \dfrac{9 \pm \sqrt{77}}{2}, \quad$ and $\quad x = \dfrac{3 \pm \sqrt{5}}{2}.$

The first two roots will be found to be reciprocals each of the other; also the second two roots.

(2) Solve the equation

$$x^5 - 3x^4 + 5x^3 - 5x^2 + 3x - 1 = 0.$$

This is of the fourth form; dividing by $x - 1$ we find the depressed equation to be

$$x^4 - 2x^3 + 3x^2 - 2x + 1 = 0.$$

This may be written

$$x^2 + 2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 1 = 0,$$

or $\quad z^2 - 2z + 1 = 0,$

whence $\quad z = 1.$

Solving the equation $x + \dfrac{1}{x} = 1$, we find

$$x = \frac{1 \pm \sqrt{-3}}{2},$$

these expressions being double roots.

Exercise 85.

Solve the equations:

1. $x^4 + 7x^3 - 7x - 1 = 0.$
2. $x^4 + 2x^3 + x^2 + 2x + 1 = 0.$
3. $x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0.$
4. $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0.$
5. $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0.$
6. $x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0.$
7. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$
8. $x^3 + mx^2 + mx + 1 = 0.$
9. $x^5 + x^4 - x^3 - x^2 + x + 1 = 0.$
10. $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$

---

507. Binomial Equations. An equation of the form

$$x^n \pm a = 0$$

is called a **binomial equation.**

We shall first consider the two equations

$$x^n - 1 = 0, \quad x^n + 1 = 0.$$

If n is *even*, the equation $x^n + 1 = 0$, by Descartes' Rule (§ 479), has no real roots; the equation $x^n - 1 = 0$ has two real roots, $+1$ and -1, the remaining $n - 2$ roots being imaginary.

If n is *odd*, the equation $x^n + 1 = 0$ has one real root, -1; the equation $x^n - 1 = 0$ has one real root, $+1$; the remaining $n - 1$ roots being in each case imaginary.

508. Now consider the equation $x^n \pm a = 0$, where a is positive. Represent by $\sqrt[n]{a}$ the real positive nth root of a. Then, if α is any root of $x^n \pm 1 = 0$, $\alpha \sqrt[n]{a}$ will be a root of $x^n \pm a = 0$.

For $$(\alpha \sqrt[n]{a})^n = \alpha^n a = \mp 1 \times a = \mp a.$$

Since α is any root of $x^n \pm 1 = 0$, the n roots of $x^n \pm a = 0$ are found by multiplying each of the n roots of $x^n \pm 1 = 0$ by $\sqrt[n]{a}$.

The roots of a binomial equation are all different. For $x^n \pm a$ and its derivative nx^{n-1}, can have no common factor involving x (§ 462).

509. *If α is a root of the equation $x^n - 1 = 0$, α^k, where k is any integer, is also a root.*

For, if α is a root, $\alpha^n = 1$.

But $(\alpha^k)^n = (\alpha^n)^k = (1)^n = 1$.

Therefore α^k is a root of $x^n = 1$, or of $x^n - 1 = 0$.

Similarly for a root of $x^n + 1 = 0$, provided k is an odd integer.

510. The Cube Roots of Unity. The equation $x^3 = 1$, or $x^3 - 1 = 0$, may be written

$$(x-1)(x^2+x+1) = 0,$$

of which we find the three roots to be:

$$1, \quad -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \quad -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

If either of the imaginary roots be represented by ω, the other is found by actual multiplication to be ω^2. This agrees with the last section.

Also, $$\omega^2 + \omega + 1 = 0.$$

In a similar manner we find the roots of $x^3 = -1$ to be

$$-1, \quad \tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}, \quad \tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3},$$

or $$-1, \quad -\omega, \quad -\omega^2.$$

511. Examples.

(1) Find the six sixth roots of 1.

We have to solve $$x^6 - 1 = 0,$$
or $$(x^3 - 1)(x^3 + 1) = 0.$$
Hence the roots are $$\pm 1, \ \pm \omega, \ \pm \omega^2.$$

(2) Find the five fifth roots of 1.

We have to solve $$x^5 - 1 = 0.$$
One root is 1; dividing by $x - 1$,
$$x^4 + x^3 + x^2 + x + 1 = 0.$$
Putting $z = x + \frac{1}{x}$, we obtain (§ 506),
$$z^2 + z - 1 = 0;$$
whence $$z = \frac{-1 \pm \sqrt{5}}{2}.$$

Solving the two quadratics $x + \frac{1}{x} = \frac{-1 \pm \sqrt{5}}{2}$, we obtain for the remaining four roots,
$$\frac{-1 + \sqrt{5} \pm \sqrt{10 + 2\sqrt{5}}\,\sqrt{-1}}{4}, \quad \frac{-1 - \sqrt{5} \pm \sqrt{10 - 2\sqrt{5}}\,\sqrt{-1}}{4}.$$

Exercise 86.

Solve the binomial equations:

1. $x^6 + 1 = 0$.
2. $x^8 - 1 = 0$.
3. $x^9 - 1 = 0$.
4. $x^5 - 243 = 0$.
5. Find the quintic on which the solution of the equation $x^{11} = 1$ depends.
6. Show that $x^3 \pm y^3 \equiv (x \pm y)(x \pm \omega y)(x \pm \omega^2 y)$.
7. Show that $x^2 + y^2 + z^2 - yz - zx - xy$
$$\equiv (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$
8. If a is an imaginary root of $x^5 - 1 = 0$, show that
$$(1 - a)(1 - a^2)(1 - a^3)(1 - a^4) = 5.$$

512. The General Cubic. We shall write the general equation of the third degree in the form

$$ax^3 + 3bx^2 + 3cx + d = 0. \qquad (1)$$

Before attempting to solve this equation we shall transform it into an equation in which the second term is wanting.

Put $z = ax + b$; $\therefore x = \frac{z-b}{a}$. Substituting this expression for x, and reducing, we obtain

$$z^3 + 3(ac - b^2)z + (a^2d - 3abc + 2b^3) = 0,$$

or, putting $H \equiv ac - b^2$, $G \equiv a^2d - 3abc + 2b^3$,

$$z^3 + 3Hz + G = 0. \qquad (2)$$

In the transformed equation put

$$z = u^{\frac{1}{3}} + v^{\frac{1}{3}}.$$

We obtain

$$(u^{\frac{1}{3}} + v^{\frac{1}{3}})^3 + 3H(u^{\frac{1}{3}} + v^{\frac{1}{3}}) + G = 0,$$

which reduces to

$$u + v + 3(u^{\frac{1}{3}}v^{\frac{1}{3}} + H)(u^{\frac{1}{3}} + v^{\frac{1}{3}}) + G = 0. \qquad (3)$$

Since we have assumed but one relation between u and v, we are at liberty to assume one more relation. Let us assume

$$u^{\frac{1}{3}}v^{\frac{1}{3}} = -H. \qquad (4)$$

Equation (3) now reduces to

$$u + v = -G. \qquad (5)$$

And (4) may be written

$$uv = -H^3. \qquad (6)$$

Eliminating v, we obtain the quadratic

$$u^2 + Gu = H^3, \qquad (7)$$

called the *reducing quadratic* of the cubic.

Solving this quadratic, we find

$$\left.\begin{aligned} u &= \frac{-G \pm \sqrt{G^2+4H^3}}{2} \\ v &= \frac{-H^3}{u} = \frac{-G \mp \sqrt{G^2+4H^3}}{2} \end{aligned}\right\}, \tag{8}$$

Since $ax+b=z=u^{\frac{1}{3}}+v^{\frac{1}{3}}$, the three values of z are

$$u^{\frac{1}{3}}-\frac{H}{u^{\frac{1}{3}}}, \qquad \omega u^{\frac{1}{3}}-\frac{H}{\omega u^{\frac{1}{3}}}, \qquad \omega^2 u^{\frac{1}{3}}-\frac{H}{\omega^2 u^{\frac{1}{3}}},$$

where $u^{\frac{1}{3}}$ is any one of the three cube roots of u.

Since there is the sign $\pm$ before the radical, we have apparently six values of z. From (4) it is seen, however, that there are really but three different values of z.

The above solution is known as *Cardan's*.

Ex. Solve, by Cardan's method,

$$2x^3-6x^2+12x-11=0.$$

Here $a=2$, $b=-2$. Putting $z=2x-2$, we obtain

$$z^3+12z-12=0.$$

$\therefore$ $H=4$, $G=-12$, and the reducing quadratic is

$$u^2-12u=64.$$

Solving, $$u=6\pm 10= 16 \text{ or } -4;$$

$$\therefore v=-\frac{H^3}{u}=-4 \text{ or } +16.$$

Hence the values of z are

$$2\sqrt[3]{2}-\sqrt[3]{4};\quad 2\omega\sqrt[3]{2}-\omega^2\sqrt[3]{4};\quad 2\omega^2\sqrt[3]{2}-\omega\sqrt[3]{4};$$

and the values of x are

$$1+\sqrt[3]{2}-\frac{1}{\sqrt[3]{2}};\quad 1+\omega\sqrt[3]{2}-\frac{\omega^2}{\sqrt[3]{2}};\quad 1+\omega^2\sqrt[3]{2}-\frac{\omega}{\sqrt[3]{2}}.$$

513. Discussion of the Solution. The above solution, while complete as an algebraic solution, is of little value in solving numerical equations.

In the case of a cubic there are three cases to consider.

I. All three roots real and unequal. In this case, $G^2 + 4H^3$ is negative (§ 503, Ex. 2), and its square root is imaginary. If we put $K^2 = -(G^2 + 4H^3)$, we shall have

$$ax + b = \left(\frac{-G + K\sqrt{-1}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - K\sqrt{-1}}{2}\right)^{\frac{1}{3}}.$$

Since there is no general algebraic rule for extracting the cube root of an imaginary expression, the case of three real and unequal roots is known as the *irreducible* case.

II. If, however, two of the roots are equal, $G^2 + 4H^3 = 0$ (§ 503, Ex. 2), and we shall have

$$ax + b = \left(\frac{-G}{2}\right)^{\frac{1}{3}} + \left(\frac{-G}{2}\right)^{\frac{1}{3}}.$$

III. If two roots are imaginary, $G^2 + 4H^3$ is positive (§ 503, Ex. 2), its square root is real, and we shall have

$$ax + b = \left(\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}}.$$

The value of the expression $G^2 + 4H^3$ determines the nature of the roots. On this account $G^2 + 4H^3$ is called the **discriminant** of the cubic.

We conclude from the above that the general solution gives us the roots of a numerical cubic in a form in which their values can be readily computed only in the second and third cases.

In either of these cases, however, the real roots are more easily found by Horner's method.

In the first case the roots may be calculated by a method involving Trigonometry. *Cf.* Chapter XXXII.

Exercise 87.

Find the three roots of:

1. $x^3 + 6x^2 = 36.$

2. $3x^3 - 6x^2 - 2 = 0.$

3. $x^3 - 3x^2 - 6x - 4 = 0.$

4. $9x^3 - 54x^2 + 90x - 50 = 0.$

5. $z^3 + 2mz^2 = m^2(m+1)^2.$

6. In the case of the cubic, putting

$$L \equiv \alpha + \omega\beta + \omega^2\gamma, \quad M \equiv \alpha + \omega^2\beta + \omega\gamma,$$

show that:

$$\begin{aligned} L^3 + M^3 &= 2\Sigma\alpha^3 - 3\Sigma\alpha^2\beta + 12\alpha\beta\gamma \\ &= -27\left(\frac{d}{a} - \frac{3bc}{a^2} + \frac{2b^3}{a^3}\right) \\ &= -\frac{27G}{a^3}. \end{aligned}$$

$$\begin{aligned} LM &= \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta \\ &= -\frac{9H}{a^2}. \end{aligned}$$

$$L^3 - M^3 = -3\sqrt{-3}(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

7. From Ex. 6, and the relation

$$(L^3 - M^3)^2 \equiv (L^3 + M^3)^2 - 4L^3M^3,$$

show that

$$a^6(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2 = -27(G^2 + 4H^3),$$

and thence deduce the conditions of § 513.

514. The General Biquadratic. We shall write the general equation of the fourth degree in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0. \qquad (1)$$

Put $z = ax + b$; $\therefore x = \frac{z - b}{a}$. Substituting this expression for x, and reducing, we obtain

$$z^4 + 6(ac - b^2)z^2 + 4(a^2d - 3abc + 2b^3)z$$
$$+ (a^3e - 4a^2bd + 6ab^2c - 3b^4) = 0. \qquad (2)$$

The fourth term may be written

$$a^2(ae - 4bd + 3c^2) - 3(ac - b^2)^2.$$

Putting, as in the case of the cubic,

$$H \equiv ac - b^2, \quad G \equiv a^2d - 3abc + 2b^3,$$

and also $\qquad I \equiv ae - 4bd + 3c^2,$

we may write (2) in the form

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0, \qquad (3)$$

in which the z^3 term is wanting.

To solve this equation, put

$$z = \sqrt{u} + \sqrt{v} + \sqrt{w}.$$

Squaring, $z^2 = u + v + w + 2(\sqrt{uv} + \sqrt{uw} + \sqrt{vw})$.

Transposing, and squaring again,

$$z^4 - 2(u + v + w)z^2 + (u + v + w)^2$$
$$= 4(uv + uw + vw) - 8z\sqrt{u}\sqrt{v}\sqrt{w}.$$

If this equation is identical with (3),

$$u + v + w = -3H,$$
$$uv + uw + vw = 3H^2 - \frac{a^2I}{4},$$
$$\sqrt{u}\sqrt{v}\sqrt{w} = -\frac{G}{2},$$

and, consequently (§ 442), u, v, and w are the roots of the cubic

$$t^3+3Ht^2+\left(3H^2-\frac{a^2I}{4}\right)t-\frac{G^2}{4}=0. \qquad (4)$$

This is known as *Euler's cubic.*

This equation may be written

$$(t+H)^3-\frac{a^2I}{4}(t+H)+\frac{a^2HI-G^2-4H^3}{4}=0.$$

Or, putting $t+H=a^2\theta$, and clearing of fractions,

$$4a^3\theta^3-Ia\theta+J=0, \qquad (5)$$

where

$$J\equiv\frac{1}{a^3}(a^2HI-G^2-4H^3)$$
$$\equiv ace+2bcd-ad^2-eb^2-c^3.$$

Equation (5) is called the *reducing cubic* of the biquadratic.

If θ_1, θ_2, θ_3, are the roots of this cubic, since $t=a^2\theta-H$, the four roots of equation (1) are given by

$$ax+b=\sqrt{a^2\theta_1-H}+\sqrt{a^2\theta_2-H}+\sqrt{a^2\theta_3-H}. \qquad (6)$$

Since each radical may be either + or −, there are apparently eight values of x obtained from the following combinations of signs:

$$\begin{array}{cccccccccccc} + & + & + & + & + & - & + & - & + & - & + & + \\ - & - & - & - & - & + & - & + & - & + & - & - \end{array}$$

But $\sqrt{u}\sqrt{v}\sqrt{w}=-\frac{G}{2}$. Consequently the number of admissible combinations is reduced to four.

The above solution is known as *Euler's.*

In determinant form

$$H\equiv\begin{vmatrix} a & b \\ b & c \end{vmatrix},\quad J\equiv\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

515. Discussion of the Solution. Represent by $\alpha, \beta, \gamma, \delta$ the roots of the given biquadratic.

Then, by equation (6), we have

$$\left.\begin{aligned} a\alpha + b &= +\sqrt{u} - \sqrt{v} - \sqrt{w}, \\ a\beta + b &= -\sqrt{u} + \sqrt{v} - \sqrt{w}, \\ a\gamma + b &= -\sqrt{u} - \sqrt{v} + \sqrt{w}, \\ a\delta + b &= +\sqrt{u} + \sqrt{v} + \sqrt{w}\,; \end{aligned}\right\} \qquad (7)$$

from which, if $\theta_1, \theta_2, \theta_3$ are the roots of the reducing cubic, we obtain

$$\left.\begin{aligned} u &= a^2\theta_1 - H = \frac{a^2}{16}(\beta + \gamma - \alpha - \delta)^2, \\ v &= a^2\theta_2 - H = \frac{a^2}{16}(\gamma + \alpha - \beta - \delta)^2, \\ w &= a^2\theta_3 - H = \frac{a^2}{16}(\alpha + \beta - \nu - \delta)^2. \end{aligned}\right\} \qquad (8)$$

There are six cases to be considered.

I. *The four roots of the biquadratic all real and unequal.*

In this case by equations (8) u, v, w, are all real. Consequently, $\theta_1, \theta_2, \theta_3$, are all real, and the cubics (4) and (6) fall under the irreducible case. (§ 513, I.)

II. *Roots all imaginary and unequal.*

By § 446 the roots must be of the forms

$$h + ki,\ h - ki,\ l + mi,\ l - mi,$$

and from equations (8)

$$u = -\frac{a^2}{4}(k - m)^2,\ v = -\frac{a^2}{4}(k + m)^2,\ w = \frac{a^2}{4}(h - l)^2.$$

So that the roots of Euler's cubic are all real, two being negative and one positive, and the cubics (4) and (6) again fall under the irreducible case. (§ 513, I.)

III. *Two roots real and two imaginary.*

In each cubic two roots are imaginary and one is real.

IV. *Two roots equal, the other two unequal.*

Each of the cubics has a pair of equal roots.

V. *Two pairs of equal roots.*

Two roots of Euler's cubic vanish, the third being $-3H$. The roots of the reducing cubic are $\frac{H}{a^2}$, $\frac{H}{a^2}$, $-\frac{2H}{a^2}$.

VI. *Three roots equal.*

The roots of Euler's cubic are $-H$, $-H$, $-H$; those of the reducing cubic all vanish.

VII. *All four roots equal.*

All the roots of both cubics vanish and $H = 0$.

516. Discriminant. Comparing the reducing cubic with the cubic

$$z^3 + 3Hz + G = 0,$$

we find the discriminant of the reducing cubic to be

$$-\frac{1}{16a^6}(I^3 - 27J^2). \qquad \text{§ 513.}$$

The expression $I^3 - 27J^2$ is called the **discriminant** of the biquadratic.

From the last section we obtain the following:

I. Discriminant of the reducing cubic *negative*; that is, $I^3 - 27J^2$ *positive.* The roots of the biquadratic are either all real or all imaginary.

II. Discriminant of the reducing cubic *vanishes*; that is, $I^3 - 27J^2 = 0$. The roots of the biquadratic fall under one of the following cases:

(a) Two roots equal, the other two unequal.

(*b*) Two pairs of equal roots. In this case $G=0$, and

$$I=\frac{12H^2}{a^2},\qquad J=\frac{8H^3}{a^3}.$$

(*c*) Three roots equal. In this case $I=0$ and $J=0$.

(*d*) Four roots equal. In this case $I=0$, $J=0$, $H=0$.

III. Discriminant of the reducing cubic *positive;* that is, I^3-27J^2 *negative.*

Two of the roots of the biquadratic are real and two are imaginary.

517. When the left-member of a biquadratic is the product of two quadratic factors with rational coefficients, the equation can be readily solved as follows:

Ex. Solve the equation

$$x^4-12x^3+12x^2+176x-96=0.$$

Here $a=1$, $b=-3$; put $z=x-3$, and we obtain

$$z^4-42z^2+32z+297=0.$$

Comparing this with

$$(z^2+pz+q)(z^2-pz+q')=0,$$

we find $\qquad q'+q-p^2=-42,\quad q'-q=\frac{32}{p},\quad qq'=297.$

Eliminating q and q', p is given by

$$p^6-84p^4+576p^2-1024=0,$$

of which two roots are found to be ± 2.

Take $p=2$, then $q'=-11$, $q=-27$, and the equation in z is

$$(z^2+2z-27)(z^2-2z-11)=0.$$

From this $\qquad z=-1\pm 2\sqrt{7},$ or $1\pm 2\sqrt{3}.$

Since $x=z+3$, we find the four values of x to be

$$2+2\sqrt{7};\ 2-2\sqrt{7};\ 4+2\sqrt{3};\ 4-2\sqrt{3}.$$

In a similar manner we can solve any biquadratic when the cubic in p^2 has a commensurable root.

Exercise 88.

Find the four roots of:

1. $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0.$

2. $x^4 - 17x^2 - 20x - 6 = 0.$

3. $x^4 - 8x^3 + 20x^2 - 16x - 21 = 0.$

4. $x^4 - 11x^3 + 46x^2 - 117x + 45 = 0.$

5. $x^4 - 7x^3 - 60x^2 + 221x - 169 = 0.$

6. Show that the biquadratic can be solved by quadratics if $G = 0$.

7. Show that the two biquadratic equations

$$ax^4 + 6cx^2 \pm 4dx + e = 0,$$

have the same reducing cubic.

8. Solve the biquadratic for the two particular cases in which $I = 0$ and $J = 0$.

9. Show that if H is positive, the biquadratic has either two or four imaginary roots.

10. Find the reducing cubic of

$$x^4 - 6ax^2 + 8x\sqrt{a^3 + b^3 + c^3 - 3abc} + (12bc - 3a^2) = 0.$$

11. Prove that J vanishes for the biquadratic

$$3a(x - 2a)^4 = 2a(x - 3a)^4.$$

12. If the roots of a biquadratic are all real, and are in harmonical progression, prove that the roots of Euler's cubic are in arithmetical progression.

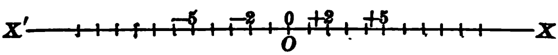
The student who wishes to pursue the subject of this chapter further is referred to Burnside and Panton's *Theory of Equations*, published by Longmans, London.

CHAPTER XXXII.

COMPLEX NUMBERS.

518. Representation of Real Numbers. Let XX' be a straight line of unlimited length. Let O be a fixed point on that line.

With any convenient unit of length measure off along the line from O to the right and left a series of equal distances.



Each of the points of division thus obtained will represent an integer (§ 8). If the points to the right represent positive integers, those to the left will represent negative integers.

The point O will represent 0.

To represent a rational fraction $\frac{a}{b}$, where a and b are integers, b being positive and a either positive or negative, we divide the unit into b equal parts, and then measure off a of these parts. The point obtained will lie between two of the points which represent integers.

We cannot find *exactly* the point which represents a given incommensurable number. We can, however, always find two fractions between which the given incommensurable number lies; and the point which represents the incommensurable number will lie between the points which represent the two fractions.

Since the difference between the fractions can be made

as small as we please, the distance between the two points representing the fractions can be made as small as we please, and the position of the point which represents the given incommensurable number can therefore be determined to any desired degree of accuracy.

It appears, then, that all real numbers may be represented by points in the line XX'.

Conversely, every point in the line XX' will represent some real number which may be integral, fractional, or incommensurable, and either positive or negative.

Instead of the representative points A, B, etc., we shall generally use the representative *lines* OA and OB.

519. Remarks on Imaginaries. Imaginary expressions are not numbers in the ordinary arithmetical sense. We perform upon them, however, the operations which we perform upon numbers, subject to the four fundamental laws which govern all algebraical operations (§ 47), viz.: the commutative, associative, distributive, and index laws. In finding the product of two imaginaries, however, the operation must be performed in a particular way (§ 168).

We shall in this chapter often extend the term *number* to include imaginary expressions.

When we are considering imaginary expressions without attempting to give them any arithmetical interpretation, there is nothing "imaginary" about the so-called imaginaries. The collection of symbols $3+4\sqrt{-1}$ is, as far as symbols go, as "real" as the collection of symbols $3+4\sqrt{2}$. It is only when we wish to obtain a result arithmetically interpretable, and arrive at an imaginary expression, that the latter can be called in a strict sense "imaginary."

On this account the term *complex number* is preferable to *imaginary number*.

520. Pure Imaginaries. A pure imaginary cannot be represented by a point on the line XX' (§ 518), since all points on that line represent real numbers. We must therefore seek elsewhere for its representative point.

Represent $\sqrt{-1}$ by i. Assuming the commutative and associative laws, we have (§ 169):

$$i \times a = ai;$$
$$i \times i \times a = i^2 \times a = (-1)a = -a;$$
$$i \times i \times i \times a = i^3 \times a = (-i)a = -ai;$$
$$i \times i \times i \times i \times a = i^4 \times a = (+1)a = +a;$$
$$i \times i \times i \times i \times i \times a = i^5 \times a = i \times a = ai;$$

and so on.

From the above we see that the effect of multiplying by i *twice* is to change a to $-a$; twice more is to change $-a$ back to $+a$. That is, *two multiplications by i reverse the sign of the multiplicand.*

Hence, two multiplications by i turn the representative line through 180°; four multiplications by i through 360°; and so on.

We may, therefore, consistently assume that *one* multiplication by i turns the representative line through 90°; three multiplications by i through 270°; and so on.

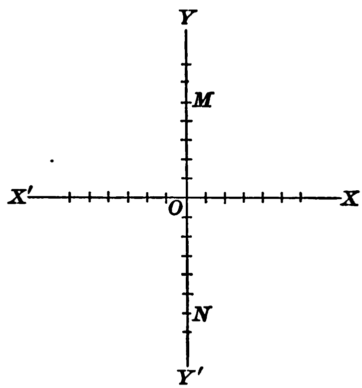
If, then, we draw through O a line YY' perpendicular to XX', all pure imaginaries will be represented by points on this line, just as all real numbers are represented by points on XX'.

521. The lines XX' and YY' are called **axes,** XX' the **axis of reals,** and YY' the **axis of pure imaginaries.** O is called the **origin.**

It is customary to regard rotation opposite to that of the hands of a clock as positive. With this convention the

point M, or the line OM, in the figure will represent $+5i$ or $+5\sqrt{-1}$; the point N, or the line ON, $-6i$ or $-6\sqrt{-1}$.

The only point which is on both axes is O. This agrees



with the fact that 0 is the only number which may be considered either real or imaginary.

Again, a and ai are measured on different lines. This agrees with the fact that a and ai are *different in kind.*

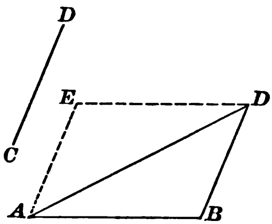
522. Vectors. A directed straight line of definite length is called a **vector.** Thus, the lines used to represent real numbers, and those used to represent pure imaginaries, are all vectors.

Vectors need not, however, be parallel to either of the axes; they may have any direction.

The line AB, considered as a vector beginning at A and ending at B, is generally written $\overline{AB}$.

Two parallel vectors which have the same length, and extend in the same direction, are said to be *equal vectors.*

523. Vector Addition. To *add* a vector CD to a vector AB we place C on B, keeping CD parallel to its original position, and draw AD.



Then, $\overline{AD} = \overline{AB} + \overline{BD}$
$= \overline{AB} + \overline{CD}.$

The addition here meant by the sign $+$ is not addition of numbers, but addition of *vectors*, generally called *geometric addition*. It is evidently identical with the composition of forces.

From the dotted lines in the figure, and the known properties of a parallelogram, it is easily seen that

$$\overline{AD} = \overline{CD} + \overline{AB}.$$

$$\therefore\ \overline{AB} + \overline{CD} = \overline{CD} + \overline{AB}.$$

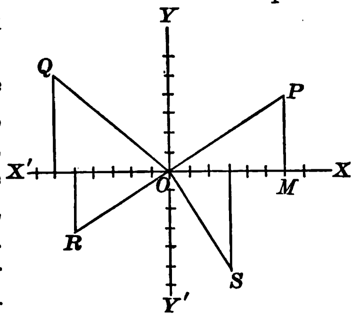
Consequently, vector addition is *commutative* (§ 21). It is easily seen that it is also *associative* (§ 27).

524. Complex Numbers. A complex number in general consists of a real part and an imaginary part, and may be written (§ 172) in the *typical form* $x + yi$, where x and y are both real.

If we understand the sign $+$ to indicate *geometric* addition, we shall obtain the vector which represents $x + yi$ as follows:

Lay off x on the axis of reals from O to M. From M draw the vector $\overline{MP}$ to represent yi. Then the vector $\overline{OP}$ is the geometric sum of the vectors $\overline{OM}$ and $\overline{MP}$, and represents the complex number $x + yi$.

Instead of the vector $\overline{OP}$ we sometimes use the point P to represent the complex number.



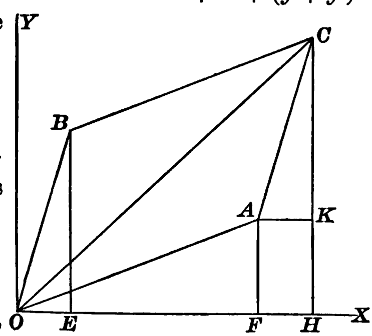
Thus, in the figure, the vectors $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, or the points P, Q, R, S, respectively, represent the complex numbers $6+4i$, $-6+5i$, $-5-3i$, $3-5i$.

In the complex number $x+yi$, x and yi are represented by vectors. Now vector addition is commutative. Consequently, $x+yi=yi+x$.

This is also evident from the preceding figure.

The expression $x+yi$ is the general expression for all numbers. This expression includes zero when $x=0$ and $y=0$; includes all real numbers when $y=0$; all pure imaginaries when $x=0$; all complex numbers when x and y both differ from 0.

525. Addition of Complex Numbers. Let $x+yi$ and $x'+y'i$ be two complex numbers. Their sum, $x+yi+x'+y'i$, may by the commutative law be written $x+x'+(y+y')i$.



Let $\overline{OA}$ and $\overline{OB}$ be the representative vectors of $x+yi$ and $x'+y'i$.

Take $\overline{AC}=\overline{OB}$;

then, $\quad \overline{OC}=\overline{OA}+\overline{OB}$.

Draw the other lines in the figure.

Then,

$$OH=OF+FH$$
$$=OF+OE=x+x',$$

and $$HC = FA + KC = FA + EB = y + y'.$$

$$\therefore\ \overline{OC} = x + x' + (y + y')i = (x + yi) + (x' + y'i).$$

But $$\overline{OC} = \overline{OA} + \overline{OB}.$$

Consequently, *the geometric sum of the vectors of two complex numbers is the vector of their sum.*

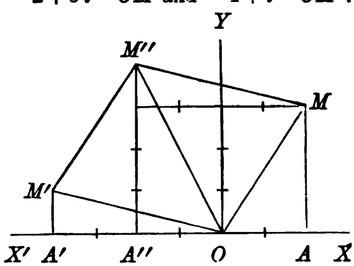
Since vector addition is commutative, it follows that the addition of complex numbers is *commutative.*

The sum of two complex numbers is the geometric sum of the sum of the real and the sum of the imaginary parts of the two numbers.

The preceding may be made clearer by a particular example.

Find the sum of $2 + 3i$ and $-4 + i$.

$2 + 3i = OM$ and $-4 + i = OM'$. If now we proceed from M, the extremity of OM, in the direction of OM' as far as the absolute value of OM', we reach the point M''.



Hence, $OM'' = -2 + 4i$, the sum of the two given complex numbers.

The same result is reached if we first find the value of $2 + (-4) = -2$. That is, if we count from O two real units to A'', and add to this sum $3i + i = 4i$; that is, count four imaginary units from A'' on the perpendicular $A''M''$.

526. Modulus and Amplitude. Any complex number, $x + yi$, can be written in the form

$$\sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}i\right).$$

The expressions $\dfrac{x}{\sqrt{x^2+y^2}}$ and $\dfrac{y}{\sqrt{x^2+y^2}}$ may be taken

as the sine and cosine of some angle ϕ, since they satisfy the equation

$$\cos^2\phi + \sin^2\phi = 1.$$

If we put $r = \sqrt{x^2+y^2}$, the complex number may be written

$$r(\cos\phi + i\sin\phi).$$

Since $r = \sqrt{x^2+y^2}$, the sign of r is indeterminate. We shall, however, take r always *positive*.

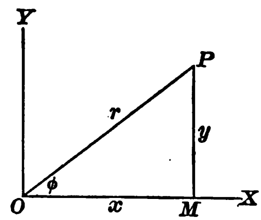
The positive number r is called the **modulus**, the angle ϕ the **amplitude**, of the complex number $x+yi$.

Let $\overline{OP}$ be the representative vector of $x+yi$. Since r is the positive value of $\sqrt{x^2+y^2}$, it is evident that r is the *length* of OP. Since

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \frac{OM}{OP},$$

and

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \frac{MP}{OP},$$



it follows that ϕ is the angle MOP.

The above is easily seen to hold true when x and y are one or both negative.

The modulus of a real number is its absolute value; the amplitude is 0 if the number is positive, 180° if the number is negative.

The modulus of a pure imaginary ai is a; the amplitude is 90° if a is positive, 270° if a is negative.

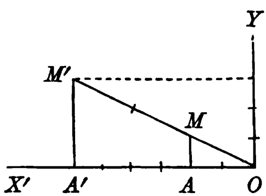
527. Since the sum of the lengths of two sides of a triangle is greater than the length of the third side, it follows from §§ 523, 525 that *the modulus of the sum of two complex numbers is less than the sum of the moduli.*

In one case, however, that in which the representative vectors are collinear, the modulus of the sum is *equal* to the sum of the moduli.

528. Multiplication by Real Numbers. Let $x + yi$ be any complex number. If the representative vector be multiplied by any real number a, it is easily seen from a figure that the product is $ax + ayi$.

Therefore, $$a(x + yi) = ax + ayi.$$

It follows that the multiplication of a complex number by a real number is *distributive*.



Ex. To multiply $-2+i$ by 3: Take $OA = -2$ on OX', and erect at A the perpendicular $AM = 1$. Then $OM = -2 + i$; and, by taking OM three times, the result is $OM' = -6 + 3i$, the product of $(-2+i)$ by 3.

529. Multiplication by Pure Imaginaries. We have seen (§ 520) that multiplying a real number, or a pure imaginary, by i turns that number through 90°. Let us consider the effect of multiplying a complex number by i.

By the commutative, associative, and distributive laws,

$$\begin{aligned} i \times r(\cos\phi + i\sin\phi) &= r(i\cos\phi - \sin\phi) \\ &= r(-\sin\phi + i\cos\phi) \\ &= r[\cos(90° + \phi) + i\sin(90° + \phi)] \end{aligned}$$

Here, also, the effect of multiplying by i is to increase ϕ to $\phi + 90°$; that is, to turn the representative vector in the positive direction through an angle of 90°.

The effect of multiplying by a pure imaginary ai will be to turn the complex number through a positive angle of 90°, and also to multiply the modulus by a.

530. Multiplication by a Complex Number. We come now to the general problem of the multiplication of one complex number by another. This includes all other cases as particular cases.

Let $r(\cos\phi + i\sin\phi)$ and $r'(\cos\phi' + i\sin\phi')$ be two complex numbers. By actual multiplication their product is

$$rr'[\cos\phi\cos\phi' - \sin\phi\sin\phi' + i(\sin\phi\cos\phi' + \cos\phi\sin\phi')].$$

By Trigonometry, this may be written

$$rr'[\cos(\phi+\phi') + i\sin(\phi+\phi')].$$

Therefore, the *modulus* of the *product* of two complex numbers is the *product* of their moduli; and the *amplitude* of the product is the *sum* of the amplitudes.

Consequently, the effect of multiplying one complex number by another is *to multiply the modulus of the first by the modulus of the second; and to turn the representative vector of the first through the amplitude of the second.*

531. Division by a Complex Number. The quotient

$$\frac{r(\cos\phi + i\sin\phi)}{r'(\cos\phi' + i\sin\phi')}$$

becomes, multiplying numerator and denominator by $\cos\phi' - i\sin\phi'$,

$$\frac{r}{r'}[\cos(\phi-\phi') + i\sin(\phi-\phi')].$$

Consequently, the *modulus* of the quotient is obtained by *dividing* the modulus of the dividend by that of the divisor; and the *amplitude* of the quotient by *subtracting* the amplitude of the divisor from that of the dividend.

532. Powers of a Complex Number. From § 530 we obtain for the case in which n is a positive integer,

$$\begin{aligned}[r(\cos\phi+i\sin\phi)]^n &= r^n[\cos(\phi+\phi+\cdots\text{ to } n \text{ terms})\\ &\quad + i\sin(\phi+\phi+\cdots\text{ to } n \text{ terms})]\\ &= r^n(\cos n\phi+i\sin n\phi).\end{aligned}$$

533. Roots of a Complex Number. From § 532, putting ϕ for $n\phi$, and r for r^n, we obtain

$$\left[\sqrt[n]{r}\left(\cos\frac{\phi}{n}+i\sin\frac{\phi}{n}\right)\right]^n = r(\cos\phi+i\sin\phi);$$

or

$$[r(\cos\phi+i\sin\phi)]^{\frac{1}{n}} = \sqrt[n]{r}\left(\cos\frac{\phi}{n}+i\sin\frac{\phi}{n}\right),$$

where by $\sqrt[n]{r}$ is meant the real positive value of the root.

The last expression gives apparently but one value for the nth root of a complex number. But we must remember that there are an unlimited number of angles which have a given sine and cosine. Thus the angles

$$\phi,\ \phi+360°,\ \phi+720°,\ \ldots\ldots,\ \phi+k(360°),$$

all have the same sine and cosine. We have, therefore, the following nth roots of $r(\cos\phi+i\sin\phi)$:

$$\sqrt[n]{r}\left(\cos\frac{\phi}{n}+i\sin\frac{\phi}{n}\right); \qquad (1)$$

$$\sqrt[n]{r}\left(\cos\frac{\phi+360°}{n}+i\sin\frac{\phi+360°}{n}\right); \qquad (2)$$

. . . .

$$\sqrt[n]{r}\left(\cos\frac{\phi+(n-1)\,360°}{n}+i\sin\frac{\phi+(n-1)\,360°}{n}\right); \quad (n)$$

$$\sqrt[n]{r}\left(\cos\frac{\phi+n\,(360°)}{n}+i\sin\frac{\phi+n\,360°}{n}\right); \qquad (n+1);$$

. . . .

In this series the $(n+1)th$ expression is the same as the *first;* the $(n+2)th$ the same as the *second;* and so on. Consequently, there are but n different nth roots, those numbered (1) to (n).

From this and the preceding section we can obtain an expression for

$$[r(\cos\phi + i\sin\phi)]^{\frac{m}{n}},$$

where $\frac{m}{n}$ is a rational fraction.

Ex. The 12 twelfth roots of 1 are:

$$\cos 0^\circ + i\sin 0^\circ = 1; \tag{1}$$

$$\cos 30^\circ + i\sin 30^\circ = \frac{\sqrt{3}+i}{2}; \tag{2}$$

$$\cos 60^\circ + i\sin 60^\circ = \frac{1+i\sqrt{3}}{2}; \tag{3}$$

$$\cos 90^\circ + i\sin 90^\circ = i; \tag{4}$$

. . . .

. . . .

$$\cos 330^\circ + i\sin 330^\circ = \frac{\sqrt{3}-i}{2}. \tag{12}$$

534. Complex Exponents. The meaning of a complex exponent is determined by subjecting it to the same operations as a real exponent.

It follows that such an expression as a^{x+yi}, where a is a real number and $x+yi$ a complex exponent, may be simplified by resolving it into two factors, one of which is a real number, and the other an imaginary power of e (§ 392).

From the ordinary rules for exponents,

$$a^{x+yi} = a^x a^{yi} = a^x(a^y)^i.$$

Put $\qquad a^y = e^u;$

then, $\qquad u = \log_e a^y = y\log_e a.$

Since $$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} \cdots\cdots \qquad (\S\ 392)$$

therefore, $$e^{ui} = 1 + ui + \frac{u^2 i^2}{\lfloor 2} + \frac{u^3 i^3}{\lfloor 3} + \frac{u^4 i^4}{\lfloor 4} + \frac{u^5 i^5}{\lfloor 5} + \cdots\cdots$$
$$= \left(1 - \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} - \cdots\cdots\right) + i\left(u - \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} - \cdots\cdots\right).$$

By the Differential Calculus it is proved that when u is the circular measure of an angle,

$$\cos u = 1 - \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} + \frac{u^6}{\lfloor 6} + \cdots\cdots, \quad \sin u = u - \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} - \cdots\cdots$$

each series being an infinite series.

Consequently, $$e^{ui} = \cos u + i \sin u,$$

and $$e^{x+ui} = e^x (\cos u + i \sin u).$$

Also, $$a^{x+yi} = a^x (\cos u + i \sin u)$$
$$= a^x [\cos (y \log_e a) + i \sin (y \log_e a)].$$

535. Trigonometric Solution of Cubic Equations. In the irreducible case (§ 513, I.) the numerical values of the roots of a cubic equation may be found by the Trigonometric tables. We have (§ 513, III.),

$$ax + b = \left(\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}} + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right)^{\frac{1}{3}}.$$

In the case to be considered $G^2 + 4H^3$ is negative (§ 513, I.).

Put $$-\frac{G}{2} = R \cos \phi, \quad \frac{\sqrt{G^2 + 4H^3}}{2} = iR \sin \phi.$$

Then, $$R^2 = (-H)^3, \quad R = (-H)^{\frac{3}{2}}.$$

And, by § 533,

$$ax + b = (-H)^{\frac{1}{2}}[\cos\phi + i\sin\phi)^{\frac{1}{3}} + (\cos\phi - i\sin\phi)^{\frac{1}{3}}].$$

The cube roots in the right number must be so taken that their product is 1, since in § 512 $u^{\frac{1}{3}}v^{\frac{1}{3}} = -H$.

The three values of $ax + b$ are:

$$2(-H)^{\frac{1}{2}}\cos\frac{\phi}{3};$$

$$2(-H)^{\frac{1}{2}}\cos\left(\frac{\phi}{3} + 120°\right);$$

$$2(-H)^{\frac{1}{2}}\cos\left(\frac{\phi}{3} + 240°\right).$$

$$\phi \text{ is given by } \tan\phi = \frac{\sqrt{-(G^2 + 4H^3)}}{G}.$$

Ex. Take the equation $z^3 - 6z + 2 = 0$.

Here $G = 2, \quad H = -2, \quad G^2 + 4H^3 = -28.$

$$\tan\phi = \frac{\sqrt{28}}{2} = \sqrt{7}. \qquad \frac{\phi}{3} = 23° 5' 54''.$$

$$\log 7 = 0.84510. \qquad \frac{\phi}{3} + 120° = 143° 5' 54''.$$

$$\log\tan\phi = 0.42255.$$

$$\phi = 69° 17' 42''. \qquad \frac{\phi}{3} + 240° = 263° 5' 54''.$$

And the three values of z are found by logarithms to be

$$2.6016, \quad -2.2618, \quad -0.3399.$$

Check:

$$\begin{array}{r} 2.6016 \\ -2.2618 \\ -0.3399 \\ \hline 0. \end{array}$$

Horner's method is, however, to be preferred to the method of the present section.

Exercise 89.

Express in the typical form:

1. $(a+bi)^4+(a-bi)^4$.

2. $\dfrac{1+i}{1+2i}+\dfrac{1-i}{1-2i}$.

3. $\dfrac{2+36i}{6+8i}+\dfrac{7-26i}{3-4i}$.

4. Show that $[(\sqrt{3}+1)+(\sqrt{3}-1)i]^3=16+16i$.

5. If $\sqrt[3]{x+yi}=a+bi$, show that
$4(a^2-b^2)=\dfrac{a}{x}+\dfrac{b}{y}$.

6. Find the modulus of $\dfrac{(3-4i)(2+3i)}{(6-4i)(15+8i)}$.

7. Find the three cube roots of $1+i$.

8. Find the five fifth roots of 1.

9. Find the four fourth roots of $3+4i$.

10. Solve the equation $z^3-12z+3=0$.

11. Solve the equation $2x^3+3x^2-3x-1=0$.

NOTE. The student who wishes to pursue the subject of this chapter further is referred to Burnside and Parton's *Theory of Equations*, and to Salmon's *Higher Algebra*.